Ann. of Appl. Math.
32:1(2016), 79-101

EVANS FUNCTIONS AND INSTABILITY OF A STANDING PULSE SOLUTION OF A NONLINEAR SYSTEM OF REACTION DIFFUSION EQUATIONS*†

Linghai Zhang‡

(*Dept. of Math., Lehigh University, 14 East Packer Avenue, Bethlehem, Pennsylvania 18015 USA*)

Abstract

In this paper, we consider a nonlinear system of reaction diffusion equations arising from mathematical neuroscience and two nonlinear scalar reaction diffusion equations under some assumptions on their coefficients.

The main purpose is to couple together linearized stability criterion (the equivalence of the nonlinear stability, the linear stability and the spectral stability of the standing pulse solutions) and Evans functions to accomplish the existence and instability of standing pulse solutions of the nonlinear system of reaction diffusion equations and the nonlinear scalar reaction diffusion equations. The Evans functions for the standing pulse solutions are constructed explicitly.

Keywords nonlinear system of reaction diffusion equations; standing pulse solutions; existence; instability; linearized stability criterion; Evans functions

2000 Mathematics Subject Classification 35Q20

1 Introduction

1.1 Mathematical Model Equations

Consider the following nonlinear system of reaction diffusion equations arising from mathematical neuroscience

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha[\beta H(u-\theta) - u] - w, \tag{1.1}$$

$$\frac{\partial w}{\partial t} = \varepsilon(u - \gamma w). \tag{1.2}$$

Also consider the following nonlinear scalar reaction diffusion equations

*This project was partly supported by a Faculty Research Grant of Lehigh University.

†Manuscript received November 3, 2015

‡Corresponding author. E-mail: liz5@lehigh.edu.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha[\beta H(u-\theta) - u], \tag{1.3}$$

and

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha[\beta H(u-\theta) - u] - w_0, \tag{1.4}$$

where $w_0 = \alpha(\beta - 2\theta) > 0$ is a positive constant. In these model equations, $u = u(x,t)$ represents the membrane potential of a neuron at position x and time t, $w = w(x,t)$ represents the leaking current, a slow process that controls the excitation of neuron membrane. The positive constants $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\varepsilon > 0$ and $\theta > 0$ represent neurobiological mechanisms. The positive constant $\theta > 0$ represents the threshold for excitation. The function $H = H(u-\theta)$ represents the Heaviside step function, which is defined by $H(u-\theta) = 0$ for all $u < \theta$, $H(0) = 1/2$ and $H(u-\theta) = 1$ for all $u > \theta$. When an action potential is generated across a neuron membrane, Na^+ ion activation is considerably faster than K^+ ion activation. The positive constant ε represents the ratio of the activation of Na^+ ion channels over the activation of K^+ ion channels. The two nonlinear scalar reaction diffusion equations may be obtained by setting $\varepsilon = 0$, $w = 0$ and $\varepsilon = 0$, $w = w_0$, respectively, in system (1.1)-(1.2). See Feroe [5-7], McKean [8-10], McKean and Moll [11], Rinzel and Keller [12], Rinzel and Terman [13], Terman [14], Wang [15] and [16] for more neurobiological backgrounds of the model system.

1.2 Main Difficulty, Main Purposes and Main Strategy

Note that there exists neither maximum principle nor conservation laws to the nonlinear system of reaction diffusion equations. The existence and instability of standing pulse solutions of the system are very important and interesting topics in applied mathematics, but they have been open for a long time, except for some numerical simulations and some claimed results without rigorous mathematical analysis. The strategy to overcome the main difficulty: coupling together linearized stability criterion and Evans functions seem to be the best way to approach the instability of the standing pulse solutions.

The main purpose of this paper is to accomplish the existence and instability of standing pulse solutions of the nonlinear system of reaction diffusion equations (1.1)-(1.2) and the nonlinear scalar reaction diffusion equation (1.3). The existence of the standing pulse solutions of both (1.1)-(1.2) and (1.3) follows from standard ideas, methods and techniques in dynamical systems. The instability of the standing pulse solutions will be accomplished by coupling together linearized stability criterion and Evans functions. The interesting and difficult points are that the eigenvalue