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GLOBAL EXISTENCE OF WEAK SOLUTIONS FOR GENERALIZED QUANTUM MHD EQUATION*

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Abstract

We prove the existence of a weak solution for a generalized quantum MHD equation in a 2-dimensional periodic box for large initial data. The existence of a global weak solution is established through a three-level approximation, energy estimates, and weak convergence for the adiabatic exponent $\gamma > 1$.

Keywords weak solutions; MHD equation; quantum hydrodynamic

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1 Introduction

The evolution of quantum MHD equations in $\Omega = T^2$ is described by the following system

$$\partial_t n + \mathrm{div}(nu) = 0, \tag{1.1a}$$

$$\partial_t(nu) + \mathrm{div}(nu \otimes u) + \nabla(P(n) + P_c(n)) - 2\mathrm{div}(\mu(n)D(u))$$
$$-\nabla(\lambda(n)\mathrm{div}u) - \frac{\hbar^2}{2} n\nabla(\varphi'(n)\Delta\varphi(n)) - (\nabla \times B) \times B = 0, \tag{1.1b}$$

$$\partial_t B - \nabla \times (u \times B) + \nabla(\nu_b(\rho)\nabla \times B) = 0, \tag{1.1c}$$

$$n(x,0) = n_0(x),\ nu(x,0) = m_0, \tag{1.1d}$$

$$B(x,0) = B_0(x),\ \mathrm{div}B_0 = 0, \tag{1.1e}$$

where the functions n, u and B represent the mass density, the velocity field and the magnetic field respectively. $P(n) = n^\gamma$ stands for the pressure, P_c is a singular continuous function and called cold pressure. $\mu(n)$ and $\lambda(n)$ denote the fluid viscosity coefficient. $\hbar > 0$ is the quantum plank constant, ν_b is the magnetic viscosity coefficient.

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Our analysis is based on the following physically grounded assumptions:

[A1] The viscosity coefficient is determined by the Newton's rheological law

$$\mu(n)=\mu_0 n^{\alpha},\quad 0<\alpha\le 1,\quad \lambda(n)=2(n\mu'(n)-\mu(n)), \tag{1.2}$$

where μ and λ are respectively the shear and bulk constant viscosity coefficients, and the dispersion term φ satisfies

$$\varphi(n)=n^{\alpha}. \tag{1.3}$$

[A2] The cold pressure P_c obeys the following growth assumption:

$$\lim_{n\to 0} P_c(n)=+\infty. \tag{1.4}$$

More precisely, we assume

$$P_c'(n)=\begin{cases} c_1 n^{-\gamma^- -1}, & n\le 1,\\ c_2 n^{\gamma-1}, & n>1, \end{cases} \tag{1.5}$$

where $\gamma^-,\gamma\ge 1,\ c_1,c_2>0$.

[A3] The positive coefficient ν_b is supposed to be a continuous function of the density, bounded from above and taking large values for small and large densities. More precisely, we assume that there exist $B>0$, positive constants d_0,d_0',d_1,d_1' large enough, $2\le a<a'<3$ and $b\in[0,\infty]$ such that

$$\text{for any } s<B,\ \ \frac{d_0}{s^a}\le \nu_b(s)\le \frac{d_0'}{s^{a'}} \quad\text{and}\quad \text{for any } s\ge B,\ \ d_1\le \nu_b(s)\le d_1' s^b. \tag{1.6}$$

Define functions $H(n)$ and $\xi(n)$ as follows:

$$\begin{cases} nH'(n)-H(n)=P(n),\quad nH_c'(n)-H_c(n)=P_c(n),\\ n\xi'(n)=\mu'(n). \end{cases} \tag{1.7}$$

The quantum fluid models have lots of applications, for instance, quantum semiconductor [6], weakly interacting Bose gases [12], superfluids [20]. More recently, dissipative quantum fluid models have been proposed by Jüngel [16], the quantum ideal magnetohydrodynamic model was derived by Hass [13]. To get the weak solution for these quantum model, it is often to introduce the damping terms $-r_0u-r_1n|u|^2u$ or the singular pressure term $P_c(n)$. These terms allow us to get the compactness of the velocity field when dealing the degenerate viscosity case. In this paper, we adopt the cold pressure form, in fact, the global existence of weak solutions can be obtained by replacing the cold pressure by a drag pressure.

There is a large amount work on the global existence of weak solutions for the compressible Navier-Stokes equation, in the constant viscosity coefficients case, one of the main result is due to P.L. Lions [18], who proved the global existence of weak solutions for the compressible Navier-Stokes system in the case of barotropic equa-