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EXISTENCE OF POSITIVE SOLUTIONS TO A SINGULAR THIRD-ORDER THREE-POINT BOUNDARY VALUE PROBLEM*†

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Abstract

In this paper, we study a singular third-order three-point boundary value problem. By using a fixed-point theorem of cone expansion-compression type, we establish results on the existence of at least one, at least two, and n positive solutions to the boundary value problem. Finally we give an example.

Keywords third-order three-point boundary value problem; positive solutions; singular nonlinearity; fixed-point theorem

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1 Introduction

Third-order differential equations arise in a variety of different areas of applied mathematics and physics, such as the deflection of a curved beam having a constant or varying cross section, three layer beam, electromagnetic waves or gravity driven flows and so on [1]. Different type of techniques have been used to study such problems: reduce them to first and/or second order equations [2], use Green's functions and comparison principles [12] (for periodic boundary value conditions), [13] (three point boundary conditions), and [14] (two point ones).

In [3], Erbe and Wang applied Krasnoselskii's work to eigenvalue problems to establish intervals of the parameter λ for which there is at least one positive solution. Many authors have used this approach or a variation to obtain eigenvalue intervals, Anderson [4], Davis and Henderson [5], but seldom used fixed-point theorem due to Leggett-Williams [6] to obtain the existence of at least three solutions to a discrete focal boundary value problem. In [7], Anderson and Avery applied a generalization of the Leggett-Williams fixed-point theorem to obtain the existence of at least three

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positive solutions to a third-order discrete focal boundary value problem. More results please see [8-11].

Motivated by the papers mentioned above, we consider the following boundary value problem (BVP)

$$\begin{cases} u'''(t) = h(t)f(t,u(t)), \quad 0 < t < 1, \\ u(0) = \delta u(\eta), \quad u'(\eta) = 0, \quad u''(1) = 0, \end{cases} \tag{1.1}$$

where $\delta \in (0,1)$, $\eta \in [\frac{1}{2}, 1)$ are constants, and the nonlinear term is allowed to be singular. Our main results show that this problem can have N positive solutions provided that the conditions on the nonlinear term on some bounded sets are appropriate, which have seldom been given in all existing literatures.

In the rest of this paper, we make the following assumptions:

(H1) $h \in C([0,1],[0,+\infty))$ and $\max\limits_{0\le t\le 1} \int_\theta^{1-\theta} G(t,s)h(s)\mathrm{d}s < \infty$, where $0<\theta<\frac{1}{2}<\eta$;

(H2) $f \in C([0,1]\times[0,+\infty),[0,+\infty))$;

(H3) there exist continuous functions $g, \lambda \in C([0,1]\times[0,+\infty),[0,+\infty))$ such that $f(t,u) \le g(t,u) + \lambda(t,u)$, $(t,u) \in (0,1)\times[0,+\infty)$;

(H4) λ is a nonincreasing function in u for any $t \in (0,1)$;

(H5) for any $r > 0$, $\max\limits_{0\le t\le 1} \int_0^1 G(t,s)h(s)\lambda(s, r\delta)\mathrm{d}s < +\infty$.

2 Preliminaries and Lemmas

Consider the Banach space $C[0,1]$ with the norm $\|u\| = \max\limits_{0\le t\le 1} |u(t)|$.

Denote

$$K = \{u \in C[0,1] : u(t) \ge \delta\|u\|, \ 0 < t < 1\},$$

$$\Omega(r) = \{u \in K, \|u\| < r\}, \quad \partial\Omega(r) = \{u \in K, \|u\| = r\}.$$

Then K is a cone of nonnegative function in $C[0,1]$.

Lemma 2.1[9] *Let $\delta \ne 0$, $h \in C[0,1]$, then the BVP*

$$\begin{cases} u'''(t) = h(t), \quad 0 < t < 1, \\ u(0) = \delta u(\eta), \quad u'(\eta) = 0, \quad u''(1) = 0 \end{cases} \tag{2.1}$$

has the unique solution $u(t) = \int_0^1 G(t,s)h(s)\mathrm{d}s$, where the Green's function $G(t,s)$ is

$$G(t,s) = \begin{cases} \dfrac{s^2}{2(1-\delta)}, & s \le t, s \le \eta, \\ -\dfrac{1}{2}t^2 + ts + \dfrac{\delta s^2}{2(1-\delta)}, & t \le s \le \eta, \\ \dfrac{1}{2}s^2 - ts + \eta t + \dfrac{\delta\eta^2}{2(1-\delta)}, & \eta \le s \le t, \\ -\dfrac{1}{2}t^2 + \eta t + \dfrac{\delta\eta^2}{2(1-\delta)}, & t \le s, \eta \le s. \end{cases} \tag{2.2}$$