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SOME PROBLEMS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS IN FIELD THEORY*

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Abstract

This paper is a brief introduction to Yang-Mills-Higgs model, Maxwell-Higgs model, Einstein's vacuum model, Yang-Baxter model, Chern-Simons-Higgs model and a discussion of the associated partial differential equation problems.

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1 Introduction

There are many interesting and challenging problems in the area of classical field theory. Classical field theory offers all types of differential equation problems which come from the two basic sets of equations in physics describing fundamental interactions, namely, the Yang-Mills equations [1,2], governing electromagnetic, weak, and strong forces, reflecting internal symmetry, the Einstein equations governing gravity, and reflecting external symmetry.

It is well known that many important physical phenomena are the consequences of various levels of symmetry breakings, internal or external, or both. These phenomena are manifested through the presence of locally concentrated solutions of the corresponding governing equations, giving rise to physical entities such as electric point charges, gravitational blackholes, cosmic strings, superconducting vortices, monopoles, dyons, and instantons. The study of these types of solutions, commonly referred to as solitons due to their particle-like behavior in interactions.

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The main purpose of this paper is to provide a quick and self-contained mathematical introduction to field theory. In particular, we shall see the origins of some important physical quantities such as energy, momentum, charges, and currents. In Section 2, we introduce the Yang-Mills theory, and give the existence of solution in different conditions. In Section 3, we discuss the existence and instability of solution of the Maxwell-Higgs equation respectively. In Section 4, we consider the solution of the initial value problem to Einstein's vacuum equation. In Sections 5 and 6, we introduce the Yang-Baxter equation and Chern-Simons-Higgs equation briefly.

2 Yang-Mills-Higgs Equation

Yang-Mills theory is a gauge theory based on the $SU(N)$ group, or more generally any compact, reductive Lie algebra. From a mathematical point of view, the gauge field is equivalent to the connection between the principal and the slave, and the material field is equivalent to the cross section of the vector cluster. The self-dual Yang-Mills equation can be deduced the integrable system.

By introducing physical variable (t,x)=(t,x_1,x_2,x_3,x_4), r=$|x|$, define derivative

$$\partial^0=\partial_t=\frac{\partial}{\partial t},\quad \partial^k=\frac{\partial}{\partial x_k},\quad \partial_r=\frac{\partial}{\partial r}=\sum_{k=1}^{3}\frac{x_k}{r}\partial^k.$$

Yang-Mills potential is $A^\mu=A^\mu(x,t)$ $(\mu=0,1,2,3)$, and the field functions are

$$\mathrm{E}^k=\partial^kA^0+\partial^0A^k+A^k\times A^0,\quad H^k=\varepsilon_{ijk}\Big(\partial^jA^i+\frac{1}{2}A^j\times A^i\Big),\quad k,i,j=1,2,3,$$

where ε_{ijk} is the convertible symbol and $\varepsilon_{123}=1$. The covariant derivative is

$$D^0=\partial^0=A^0x,\quad D^k=\partial^k+A^kx,\quad k=1,2,3.$$

For three-dimensional space vector $\phi=\phi(x,t)$ (Higgs field), set $\psi^\mu=D^\mu\phi$, then Lagrange density [3] is

$$\mathcal{L}=\frac{1}{2}\Bigg\{\sum_{k=1}^{3}|\mathrm{E}^k|^2-\sum_{k=1}^{3}|H^k|^2-m^2\Bigg(|A^0|^2-\sum_{k=1}^{3}|A^k|^2\Bigg)+|D^0\phi|^2-\sum_{k=1}^{3}|D^k\phi|^2\Bigg\}-V(\phi),\tag{2.1}$$

where $V(\phi)=V_0(|\phi|^2)$, V_0 is a real function of real variable. Let $V'(\phi)=2\phi V_0'(|\phi|^2)$, where V_0' is the derivative of V, the motion equations of $\mathcal{L}$ has the following form:

$$\begin{cases} D^0\mathrm{E}^i=\varepsilon_{ijk}D^jH^k-m^2A^i+\psi^i\times\phi,\quad i=1,2,3,\\ \displaystyle\sum_{k=1}^{3}D^kH^k=\psi^0\times\phi+m^2A^0,\\ \displaystyle D^0\psi^0-\sum_{k=1}^{3}D^k\psi^k=-V'(\phi).\end{cases}\tag{2.2}$$