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ASYMPTOTIC EIGENVALUE ESTIMATION FOR A CLASS OF STRUCTURED MATRICES*†

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Abstract

In this paper we consider eigenvalue asymptotic estimations for a class of structured matrices arising from statistical applications. The asymptotic upper bounds of the largest eigenvalue ($\lambda_{\max}$) and the sum of squares of eigenvalues $\left(\sum\limits_{i=1}^{n} \lambda_i^2\right)$ are derived. Both these bounds are useful in examining the stability of certain Markov process. Numerical examples are provided to illustrate tightness of the bounds.

Keywords Toeplitz matrix; eigenvalue; rank-one modification; trace

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1 Introduction

The necessity of analyzing asymptotic behavior of eigenvalue distribution of structure matrices, e.g., Toeplitz matrices, block Toeplitz matrices and Toeplitz like matrices, arise in many fields of applications [1-3]. The eigenvalue distribution of sequences of symmetric Toeplitz matrices was revealed [1]. As an application, these results have been applied to the study of covariance matrices of a second order

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stationary process. In more recent research, extensions have been made to block Toeplitz matrices with non-Toeplitz blocks [4,5]. In this paper, we consider a class of structure matrices arising in the study of order statistics under exponentiality. The matrix $A \in \mathcal{R}^{n\times n}$ is structured with its entries be defined by

$$a_{ij} = \mathrm{e}^{-\frac{|i-j|}{n}P} - \mathrm{e}^{-\frac{i+j}{n}P} \tag{1}$$

where P is a parameter with an order normally less than $n^{\frac{1}{2}}$, $n \to +\infty$.

Let λ_i, $i = 1, 2, \cdots, n$, be eigenvalues of A, and $\lambda_{\max}$ be the largest eigenvalue. Then $\lambda_{\max}$ is the optimal constant of certain logarithmic Sobolev inequality, and $\sum\limits_{i=1}^{n} \lambda_i^2$ is a useful quantity in related applications. Therefore, upper bounds of $\lambda_{\max}$ and $\sum\limits_{i=1}^{n} \lambda_i^2$ (as $n \to \infty$) are important in determining asymptotic stability of the process. In this note, we intend to give proper upper bounds for $\lambda_{\max}$ and $\sum\limits_{i=1}^{n} \lambda_i^2$ respectively, as $n \to \infty$.

By the definition of matrix A, it is easy to see that A is a positive matrix [6]. For simplicity, we denote positive matrix A by $A > 0$ and the nonnegative matrix A by $A \geq 0$. The following results are useful in deriving the main results of the paper. For completeness, we list them as follows.

Lemma 1[7] *Let $B \geq 0$ be an irreducible matrix. Then:*

(1) *The matrix B has a positive real eigenvalue equal to its spectral radius;*

(2) *the spectral radius $\rho(B)$ is a simple eigenvalue of B;*

(3) *there is an eigenvector $x > 0$ that is corresponding to eigenvalue $\rho(B)$;*

(4) *$\rho(B)$ increases when any entry of B increases.*

Lemma 2[8] *The positive matrix is irreducible.*

This paper is organized as follows. In Section 2, by regarding matrix A as a rank-one modification of Toeplitz matrix, we derive the asymptotic upper bound for the largest eigenvalue. In Section 3, by computing the trace of A^2, we obtain the asymptotic upper bound for the sum of squares of eigenvalues $\left(\sum\limits_{i=1}^{n} \lambda_i^2\right)$. Finally, we conclude the paper in Section 4.

2 Asymptotic Upper Bound of the Largest Eigenvalue

Define $\widetilde{A} = (\widetilde{a}_{i,j})_{n\times n}$ by

$$\widetilde{a}_{ij} = \mathrm{e}^{-\frac{|i-j|}{n}P}. \tag{2}$$

Then

$$A = \widetilde{A} - uu^{\mathrm{T}}, \tag{3}$$

where $u = [\mathrm{e}^{\frac{-1}{n}P}, \cdots, \mathrm{e}^{\frac{-n}{n}P}]^{\mathrm{T}}$, and the superscript $^{\mathrm{T}}$ denotes the transpose of a vector or matrix. Since matrix $\widetilde{A}$ has Toeplitz structure, it follows that the original