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ON THE WELL-POSEDNESS OF WEAK SOLUTIONS*†

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Dedicated to the 90th Birthday of Xiaqi Ding

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Abstract

This is to comment on the well-posedness of weak solutions for the initial value problem for partial differential equations. In recent decades, and particularly in recent years, there have been substantial progresses on construction by convex integration for the study of non-uniqueness of solutions for incompressible Euler equations, and even for compressible Euler equations. This prompts the question of whether it is possible to give a sense of well-posedness, which is narrower than the canonical Hadamard sense, so that the evolutionary equations are well-posed. We give a brief and partial review of the related results and offer some thoughts on this fundamental topic.

Keywords nonlinear partial differential equations; well-posedness theory; weak solutions

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1 Introduction

The canonical Hadamard well-posedness theory for the initial value problem for evolutionary partial differential equations requires the existence, uniqueness and continuous dependence of the solution on its initial data with respect to a given topological space. Suppose that the solution space is set to be $L_2(\boldsymbol{x})$. Then the solution is found and unique in $L_2(\boldsymbol{x})$ and depends continuously on its initial value in the $L_2(\boldsymbol{x})$ topology.

The well-posedness of classical solutions follows basically from the mean value theorem, as the smoothness of the solutions allows for the classical application of calculus. For linear equations, the well-posedness of weak solutions is shown by the a priori estimates, as a consequence of the principle of linear superposition. There is

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usually a problem with the well-posedness of weak solutions for nonlinear equations. This is because the usual compactness argument in constructing the weak solutions does not yield sufficient structure to the solutions.

In Section 2, we give a brief review of the non-uniqueness of weak solutions. In Section 3, the important well-posed theory for shock wave theory is described. The last two sections are about the viscous conservation laws, with Section 4 reviewing known results and Section 5 describing a well-posedness theory for weak solutions of the Navier-Stokes equations in gas dynamics, done recently by Shih-Hsien Yu and the author.

2 Weak Solutions

Weak solutions are constructed using various versions of compactness argument. There is the well-known Leray's weak solutions for the incompressible Navier-Stokes equations [17]. Another interesting example is the weak solutions for the Boltzmann equation in the kinetic theory constructed by DiPerna and Lions [7]. Compactness arguments yield not sufficient control over the evolution of the solutions in time. This makes it difficult for showing the well-posedness of weak solutions. For most situations, it is not known if the problem is well-posed. However, in some cases, the problem is shown to be ill-posed. Scheffer [26] constructed weak solutions for the the incompressible Euler equations which have compact support in space and time. As the zero solution is a trivial solution, this gives a striking non-uniqueness of weak solutions for the incompressible Euler equations. The method of convex integration has been greatly improvised, and there have been very substantial progresses in this important research direction. For instance, there is a non-uniqueness of weak solutions for the compressible Euler equations, De Lellis and Szkelyhidi [4].

We note in passing that the non-uniqueness construction of weak solutions for the compressible Euler equations does not extend to the potential flow equation. The potential flow equation possesses shock waves, but not vortex. Thus the extension from incompressible to compressible Euler equations seems to concern mostly the vortex, and not the acoustic waves. Thus the non-uniqueness for the compressible Euler equations does not point directly to the deficiency of the entropy condition for the shock waves [16, 19].

3 System of Hyperbolic Conservation Laws

We turn our attention to the shock wave theory for system of hyperbolic conservation laws. The solutions are in general weak solutions containing shock waves. There is an important well-posedness theory for weak solutions for system of hyperbolic conservation laws

$$\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = 0, \quad \boldsymbol{u} \in \mathbb{R}^n.$$