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PROPERTIES OF SOLUTIONS OF n-DIMENSIONAL INCOMPRESSIBLE NAVIER-STOKES EQUATIONS*

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Abstract

Consider the n-dimensional incompressible Navier-Stokes equations

$$\frac{\partial}{\partial t}\mathbf{u} - \alpha\triangle\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}(\mathbf{x}, t), \quad \nabla\cdot\mathbf{u} = 0, \nabla\cdot\mathbf{f} = 0,$$
$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \nabla\cdot\mathbf{u}_0 = 0.$$

There exists a global weak solution under some assumptions on the initial function and the external force. It is well known that the global weak solutions become sufficiently small and smooth after a long time. Here are several very interesting questions about the global weak solutions of the Cauchy problems for the n-dimensional incompressible Navier-Stokes equations.

- Can we establish better decay estimates with sharp rates not only for the global weak solutions but also for all order derivatives of the global weak solutions?
- Can we accomplish the exact limits of all order derivatives of the global weak solutions in terms of the given information?
- Can we use the global smooth solution of the linear heat equation, with the same initial function and the external force, to approximate the global weak solutions of the Navier-Stokes equations?
- If we drop the nonlinear terms in the Navier-Stokes equations, will the exact limits reduce to the exact limits of the solutions of the linear heat equation?
- Will the exact limits of the derivatives of the global weak solutions of the Navier-Stokes equations and the exact limits of the derivatives of the global smooth solution of the heat equation increase at the same rate as the order m of the derivative increases? In another word, will the ratio of the exact limits for the derivatives of the global weak solutions of the Navier-Stokes equations be the same as the ratio of the exact limits for the derivatives of the global smooth solutions for the linear heat equation?

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The positive solutions to these questions obtained in this paper will definitely help us to better understand the properties of the global weak solutions of the incompressible Navier-Stokes equations and hopefully to discover new special structures of the Navier-Stokes equations.

Keywords the n-dimensional incompressible Navier-Stokes equations; decay estimates with sharp rates; exact limits; appropriate coupling of existing ideas and results; Fourier transformation; Parseval's identity; Lebesgue's dominated convergence theorem; Gagliardo-Nirenberg's interpolation inequality

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1 Introduction

1.1 Mathematical model equations

Consider the Cauchy problem for the n-dimensional incompressible Navier-Stokes equations

$$\frac{\partial}{\partial t}\mathbf{u} - \alpha\triangle\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}(\mathbf{x},t), \quad \nabla\cdot\mathbf{u} = 0, \nabla\cdot\mathbf{f} = 0, \tag{1}$$

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \quad \nabla\cdot\mathbf{u}_0 = 0. \tag{2}$$

The real vector valued function $\mathbf{u} = \mathbf{u}(\mathbf{x},t)$ represents the velocity of the fluid at position $\mathbf{x}$ and time t. The real scalar function $p = p(\mathbf{x},t)$ represents the pressure of the fluid at $\mathbf{x}$ and t. The positive constant $\alpha > 0$ represents the diffusion coefficient. See Leray [7], Temam [14] and [15].

Consider the Cauchy problem for the linear heat equation

$$\frac{\partial}{\partial t}\mathbf{v} - \alpha\triangle\mathbf{v} = \mathbf{f}(\mathbf{x},t), \quad \nabla\cdot\mathbf{v} = 0, \nabla\cdot\mathbf{f} = 0, \tag{3}$$

$$\mathbf{v}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \quad \nabla\cdot\mathbf{u}_0 = 0. \tag{4}$$

Can we use the solution of the linear heat equation to approximate the solution of the Navier-Stokes equation? Theorems 2 and 4 given in Subsection 1.5 not only provide a positive solution but also demonstrate how well it approximates.

1.2 Previous related results

Let us review several well known results about the n-dimensional incompressible Navier-Stokes equations (1)-(2).

The existence and regularity of the global weak solutions: First of all, let us consider the case $n = 2$. If the initial function $\mathbf{u}_0 \in \mathcal{S}(\mathbb{R}^2)$ and the external force $\mathbf{f} \in L^1(\mathbb{R}^+, L^2(\mathbb{R}^2)) \cap C^\infty(\mathbb{R}^n \times \mathbb{R}^+)$, then there exists a unique global smooth solution

$$\mathbf{u} \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+).$$

See Leray [7], Temam [14] and [15]. Secondly, let us consider the case $n \geq 3$ (basically, n=3 or n=4). Suppose that the initial function $\mathbf{u}_0 \in L^2(\mathbb{R}^n)$ and the external