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A NOTE ON CLINE'S FORMULA FOR TWO SUBCLASSES OF GENERALIZED DRAZIN INVERSES*†

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Abstract

For a Banach algebra $\mathcal{A}$ with identity and $a, b, c, d \in \mathcal{A}$, the relations between the extended g-Drazin inverse (resp. generalized strong Drazin inverse) of ac and that of bd are given, when $bac = bdb$ and $cac = cdb$.

Keywords generalized Drazin inverse; extended g-Drazin inverse; generalized strong Drazin inverse; Cline's formula

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1 Introduction and preliminaries

Let $\mathcal{A}$ be a Banach algebra with identity. For $a \in \mathcal{A}$, the notations $\sigma(a)$, $acc\sigma(a)$ and $r(a)$ denote the spectrum, accumulated spectral points and the spectral radius of a, respectively. The sets of all invertible elements, nilpotent elements and quasinilpotent elements of $\mathcal{A}$ are denoted by $\mathcal{A}^{inv}$, $\mathcal{A}^{nil}$ and $\mathcal{A}^{qnil}$, respectively. Set

$$\mathcal{A}^{qnil} = \{a \in \mathcal{A} : \sigma(a) = \{0\}\};$$

$$\triangle_1 = \{a \in \mathcal{A} : \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = 0\};$$

$$\triangle_2 = \{a \in \mathcal{A} : 1 - ax \in \mathcal{A}^{inv} \text{ for any } x \text{ commuting with } a\}.$$

Lemma 1.1[5] *Let $\mathcal{A}$ be a Banach algebra with identity. Then $\mathcal{A}^{qnil} = \triangle_1 = \triangle_2$.*

In 1958, Drazin [3] introduced a new notion of invertibility, which is now commonly known as Drazin invertibility, in semigroup. The Drazin inverse plays a significant role in operator theory, singular differential equations, Markov chains, etc., see [1]. Recall that an element $a \in \mathcal{A}$ is Drazin invertible if there exists an element $x \in \mathcal{A}$ such that $ax = xa$, $xax = x$ and $a - a^2x \in \mathcal{A}^{nil}$. The nilpotent index

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of $a-a^2x$ is called the Drazin index of a. We use a^D to denote the Drazin inverses of a. By extending nilpotent element to quasinilpotent element, Koliha [6] introduced a concept of generalized Drazin invertibility in Banach algebras.

Definition 1.1[6] An element $a \in \mathcal{A}$ is generalized Drazin invertible if there exists an element $x \in \mathcal{A}$ such that

$$xa = ax, \quad xax = x \text{ and } a - a^2x \in \mathcal{A}^{qnil}.$$

In this case, x is called the generalized Drazin inverse of a and is denoted by a^{gD}.

If such x exists, it is unique. The notation $\mathcal{A}^{gD}$ denotes the set of all generalized Drazin invertible elements in $\mathcal{A}$. It is easy to see that $\mathcal{A}^{qnil} \subseteq \mathcal{A}^{gD}$, because the generalized Drazin inverse of a quasinilpotent element is zero. The generalized Drazin spectrum is defined by

$$\sigma_{gD} = \{\lambda \in \mathbb{C} : \lambda - a \notin \mathcal{A}^{gD}\}.$$

In [6], Koliha proved that $a \in \mathcal{A}^{gD}$ if and only if $0 \notin acc\sigma(a)$. This implies $\sigma_{gD}(a) = acc\sigma(a)$. In order to generalize the ideal of Koliha, Mosić [9] introduced a new type of outer generalized inverse.

Definition 1.2[9] An element $a \in \mathcal{A}$ is extended g-Drazin invertible (or eg-Drazin invertible) if there exists an element $x \in \mathcal{A}$ such that

$$xa = ax, \quad xax = x \text{ and } a - a^2x \in \mathcal{A}^{gD}.$$

In this case, x is called the extended g-Drazin inverse of a and is designated by a^{ed}.

Different from generalized Drazin inverse, the eg-Drazin inverse is not unique, see [9, Example 2.1]. However, the existence of extended g-Drazin inverse and generalized Drazin inverse are equivalent. We use $\mathcal{A}^{ed}$ to denote the set of all generalized Drazin invertible elements in $\mathcal{A}$. By Lemma 1.2, we have $\mathcal{A}^{ed} = \mathcal{A}^{gD}$.

Lemma 1.2[8] *Let $a \in \mathcal{A}$. The following statements are equivalent:*

(1) *a is eg-Drazin invertible;*

(2) *a is generalized Drazin invertible;*

(3) $0 \notin \sigma_{gD}(a)(= acc\sigma(a))$*;*

(4) *there exists an idempotent $q \in \mathcal{A}$ such that*

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_q,$$

where $a_1 \in (q\mathcal{A}a)^{-1}$ and $a_2 \in \mathcal{A}^{gD}$.

Motivated by the strong nil-cleanness, the notion of strong Drazin inverse in a ring was introduced and studied in [11]. By extending nilpotent element to quasinilpotent element, the generalized strong Drazin inverse was introduced in Banach algebras. More interesting properties of strong Drazin inverse and generalized