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BASIC THEORY OF GENERALIZED p-TYPE RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS*

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Abstract

A new general class of retarded functional differential equations (that is, RFDEs) with unbounded delay and with finite memory is introduced. The basic theories of existence, uniqueness, continuation, and continuous dependence are developed.

Keywords p-RFDEs; existence; uniqueness; continuation; continuous dependence on initial value

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1 Introduction

The pioneering work of Krasovskiĭ [9, 10] clarified the functional nature of systems with delays. The functional approach provides a foundation for the theory of differential equations with delays and the term functional differential equation (that is, FDE) is used as a synonym for systems with delays. In the framework of functional approach, different aspects of the theory of FDEs have been well developed and the systematic presentation of these results can be found in a number of excellent books [5, 7, 8, 11, 15, 18, 21].

In the classical theory of retarded functional differential equation (that is, RFDEs), the notation x_t defined by $x_t(\theta) = x(t + \theta)$, expressing "taking into account" the history of the process $x(t)$ considered, is the modern one introduced by Hale [6] and essentially due to Krasovskiĭ [9]. The basic theory of RFDEs has been established by many researchers. The description and classification of RFDEs are traditionally

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discussed by types of time-delay: finite/bounded delay, unbounded delay but finite memory, and infinite delay [12, 13, 21]. The RFDEs with a general delay structure in a unified way were introduced from a dynamical systems viewpoint [14] based on the notation defined by Walther in [17], and hence some RFDEs with unbounded time- and state-dependent delays could be applied.

As a cornerstone of the basic theory, the unified description of RFDEs has been studied a lot. For the systems of RFDE with bounded delay and infinite delay, the unified description of state variables with delay terms can be understood easily which are presented in many monographs and papers. For the case with unbounded delay but finite memory, it can be generalized by p function, hence the corresponding system can be uniformly expressed in a general form called p-type retarded functional differential equation (p-RFDE) [12, 13, 20].

We now recall some basic notions for the so-called p-function and nonlinear systems of RFDE with unbounded delay but with finite memory in the sense given in [12, 20].

Definition 1.1 For $r > 0$, the function $p \in C(\mathbb{R} \times [-r, 0], \mathbb{R})$ is said to be a p-function if it has the following properties:

(i) $p(t, 0) = t$;

(ii) $p(t, -r)$ is a nondecreasing function of t;

(iii) there exists a $\sigma \geq -\infty$ such that $p(t, \theta)$ is an increasing function for θ for each $t \in (\sigma, \infty)$;

(iv) $p(t, 0) - p(t, -r) > 0$ for $t \in (\sigma, \infty)$.

Let $t_0 \geq 0$, $A > 0$ and $x \in C([p(t_0, -r), t_0 + A], \mathbb{R}^n)$. For any $t \in [t_0, t_0 + A]$, with the aid of p-function, the symbol x_t is defined as

$$x_t(\theta) = x(p(t, \theta)), \quad -r \leq \theta \leq 0,$$

so that $x_t \in \mathscr{C} := C([-r, 0], \mathbb{R}^n)$. Consider the system

$$x'(t) = f(t, x_t), \tag{1.1}$$

where $f \in C(\mathbb{R}^+ \times \mathscr{C}, \mathbb{R}^n)$, which is called the system of p-type RFDE (that is, p-RFDE).

The p-RFDEs have been well developed, e.g., the basic theories were established in [12, 19, 20, 22, 23] and the existence of positive solution was explored in [1–3, 16]. Note that, in Definition 1.1, the conditions (i) and (iii) imply property (iv) and hence (iv) can be removed in the definition of p-function. Even though, the p-function still has great limitations so that many RFDEs in very simple form can not be classified into p-RFDE such as

$$\dot{x}(t) = f(t, x(t), x(t - t^2)), \quad t \geq \sigma > 0.$$

Motivated by the above facts, the principal objective of the paper is to generalize