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Convergence of an Embedded Exponential-Type Low-Regularity Integrators for the KdV Equation without Loss of Regularity

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Abstract. In this paper, we study the convergence rate of an Embedded exponential-type low-regularity integrator (ELRI) for the Korteweg-de Vries equation. We develop some new harmonic analysis techniques to handle the "stability" issue. In particular, we use a new stability estimate which allows us to avoid the use of the fractional Leibniz inequality,

$$\left|\left\langle J^{\gamma}\partial_x(fg), J^{\gamma}f\right\rangle\right| \lesssim \|f\|_{H^\gamma}^2 \|g\|_{H^{\gamma+1}},$$

and replace it by suitable inequalities without loss of regularity. Based on these techniques, we prove that the ELRI scheme proposed in [41] provides $\frac{1}{2}$-order convergence accuracy in H^γ for any initial data belonging to H^γ with $\gamma > \frac{3}{2}$, which does not require any additional derivative assumptions.

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1 Introduction

The Korteweg-de Vries (KdV) equation arises as a model equation from the weakly nonlinear long waves and describes the propagation of shallow water waves in a channel [23]. It has taken a wide range of applications in a diverse field of the industries, especially in terms of application and technology. In this paper, we consider the KdV equation with periodic boundary conditions

$$\begin{cases} \partial_t u(t,x)+\partial_x^3 u(t,x)=\dfrac{1}{2}\partial_x(u(t,x))^2, & t>0, \quad x\in\mathbb{T}, \\ u(0,x)=u_0(x), & x\in\mathbb{T}, \end{cases} \tag{1.1}$$

where $\mathbb{T}=(0,2\pi)$, $u=u(t,x):\mathbb{R}^+\times\mathbb{T}\to\mathbb{R}$ is the unknown and $u_0\in H^{s_0}(\mathbb{T})$ with some $0\le s_0<\infty$ is a given initial data.

Many authors have studied the initial value problem of the KdV equation both on the real line and in the period case, and established the global well-posedness in H^s for $s\ge -1$; see [4, 18, 20]. The numerical solution of the KdV equation has been important in a wide range of fields. One interesting question in the numerical solution of the KdV equation is how much regularity is required in order to have certain desired convergence rates. Correspondingly, many numerical methods and numerical analysis were developed to address this question, including finite difference methods [5, 17, 21, 37], finite element methods [1, 6, 38], operator splitting [14–16, 36], spectral methods [7, 28, 29, 35], discontinuous Galerkin methods [26, 42] and exponential integrators [2, 11, 12].

Among the many numerical time integration methods for time-dependent partial differential equations (PDEs), the splitting methods are very popular in many classic studies. We refer the readers to [8, 13, 30] for an extensive overview of splitting methods. As far as we know, operator splitting methods for the KdV equation (often referred to as fractional-step methods) first appeared in [36] and were analysed rigorously in [15]. Operator splitting methods have been developed into a systematic approach for constructing time-stepping methods for evolutionary PDEs. In particular, Holden et al. [14, 16] proved that the Godunov and Strang splitting methods for the KdV equation converge with the first-order and the second-order rates in H^γ with $\gamma\ge 1$, if the initial datum belong to $H^{\gamma+3}$ and $H^{\gamma+5}$, respectively. For the nonlinear Schrödinger equation (NLS), Lubich [27] proved that for the initial data in H^4, the Strang splitting scheme provides the first-order and the second-order convergence in H^2 and L^2, respectively. In addition to the splitting method, exponential integrators is also a very effective numerical method for solving partial differential equations including hyperbolic and parabolic problems [9, 10]. In particular, Hochbruck and Ostermann [11] presented some typical applications that