Ann. Appl. Math.
doi: 10.4208/aam.OA-2020-0003

Vol. **37**, No. 1, pp. 31-60
February 2021

Stability of the Semi-Implicit Method for the Cahn-Hilliard Equation with Logarithmic Potentials

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Received 11 October 2020; Accepted (in revised version) 15 February 2021

Abstract. We consider the two-dimensional Cahn-Hilliard equation with logarithmic potentials and periodic boundary conditions. We employ the standard semi-implicit numerical scheme, which treats the linear fourth-order dissipation term implicitly and the nonlinear term explicitly. Under natural constraints on the time step we prove strict phase separation and energy stability of the semi-implicit scheme. This appears to be the first rigorous result for the semi-implicit discretization of the Cahn-Hilliard equation with singular potentials.

AMS subject classifications: 65M60, 35Q35

Key words: Cahn-Hilliard equation, logarithmic kernel, semi-implicit scheme, energy stability.

1 Introduction

Consider the 2D Cahn-Hilliard equation on $\Omega=\mathbb{T}^2=[-\pi,\pi)^2$:

$$\begin{cases}\partial_t u=\Delta\mu=\Delta(-\nu\Delta u+F'(u)), & (t,x)\in(0,\infty)\times\Omega,\\ u\big|_{t=0}=u_0,\end{cases} \tag{1.1}$$

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where $u:\Omega\to(-1,1)$ is the order parameter of a two-phase system such as a binary alloy, and the term μ denotes the chemical potential. The two end-points $u=\pm1$ correspond to pure states. The coefficient $\nu>0$ denotes mobility. In this paper we take it to be a constant parameter. The thermodynamic potential $F:(-1,1)\to\mathbb{R}$ is given by

$$F(u)=\frac{\theta}{2}\Big((1+u)\ln(1+u)+(1-u)\ln(1-u)\Big)-\frac{\theta_c}{2}u^2,\quad 0<\theta<\theta_c, \tag{1.2a}$$

$$f(u)=F'(u)=-\theta_c u+\frac{\theta}{2}\ln\frac{1+u}{1-u}=:-\theta_c u+\tilde{f}(u),\qquad F''(u)=\frac{\theta}{1-u^2}-\theta_c, \tag{1.2b}$$

where the logarithmic part accounts for the entropy of mixing. The parameters θ and θ_c corresponds to the absolute temperature and the critical temperature respectively. Denote by $u_+>0$ the positive root of the equation $f(u)=0$ (see (1.2b)). Under the condition $0<\theta<\theta_c$ the potential F takes the form of a double-well with two equal minima at u_+ and $-u_+$ which are usually called binodal points. One should note that the condition $0<\theta<\theta_c$ is of physical importance since it guarantees that that F has a double-well form and phase separation can indeed occur. For $u_s=(1-\theta/\theta_c)^{\frac{1}{2}}$, the region $(-u_s,u_s)$, where $F''(u)<0$ is called the spinodal interval. If the quenching is shallow, i.e., the temperature θ is close to the absolute temperature θ_c, then one can expand near $u=0$ and obtain the usual quartic polynomial approximation of the free energy.

The usual energy conservation takes the form:

$$\frac{d}{dt}\mathcal{E}(u)=-\||\nabla|^{-1}\partial_t u\|_2^2,\qquad \mathcal{E}(u)=\int_\Omega\Big(\frac{1}{2}\nu|\nabla u|^2+F(u)\Big)dx. \tag{1.3}$$

Note that for $u\in(-1,1)$, the term $F(u)$ is bounded by an absolute constant, and the only coercive quantity in $\mathcal{E}(u)$ is the gradient term.

Remark 1.1. We note that the usual quartic polynomial approximation of the free energy $F(u)$ is given by (below the series converges for $u\in[-1,1]$)

$$\begin{aligned}F(u)&=-\frac{\theta_c}{2}u^2+\theta\sum_{k=0}^{\infty}\frac{u^{2k+2}}{(2k+1)(2k+2)}\\&\approx F_{\text{quartic}}(u)=\frac{\theta}{2}\cdot\frac{u^4}{6}+\Big(\frac{\theta}{2}-\frac{\theta_c}{2}\Big)u^2.\end{aligned}$$

The standard double-well potential $\text{const}\cdot(u^2-1)^2$ corresponds to the specific choice $\theta/\theta_c=3/4$. However, this approximation introduces a nontrivial shift of the location of the minimum. Namely for the original free energy $F(u)$, its two equal minima