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Transverse Instability of the CH-KP-I Equation

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Abstract. The Camassa–Holm–Kadomtsev–Petviashvili-I equation (CH-KP-I) is a two dimensional generalization of the Camassa–Holm equation (CH). In this paper, we prove transverse instability of the line solitary waves under periodic transverse perturbations. The proof is based on the framework of [18]. Due to the high nonlinearity, our proof requires necessary modification. Specifically, we first establish the linear instability of the line solitary waves. Then through an approximation procedure, we prove that the linear effect actually dominates the nonlinear behavior.

AMS subject classifications: 35B35, 35C07, 35G25

Key words: Camassa-Holm-Kadomtsev-Ketviashvili-I equation, line solitary waves, transverse instability.

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1 Introduction

Surface water wave is too much of a monster to tame. Thus various asymptotic models have been developed to simplify it. In the realm of shallow water waves, these models include the KdV equation [14], the Camassa–Holm equation [4,7], etc.. They are all unidirectional approximation models, which means that we assume the surface elevation is uniform in the transverse direction. A key observation is that these models all admit Hamiltonian structure, which indicates that it is reasonable to expect a systematic way to deal with a class of problems based on that structure.

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One problem focuses on the orbital stability around solitary waves–traveling waves which decay to zero at infinity. Roughly speaking, we want to know if the solution consistently stays in the neighborhood of a solitary wave and its translation when its initial data does. A naive thinking why it is true is that the solitary wave holds the least Lagrangian action energy, so the object around it is "willing" to evolve like that. One of the universal treatments is by center manifold theory. The center manifold theory is an equivalent but more algebraic form of the original problem (e.g., under Fourier transform), based on spectral decomposition. The "finite dimension" version of the spectral decomposition is purely algebraic in taste, while its corresponding "infinite" counterpart has topology coming into play as a role of approximation to mimic the world of "finite". This thought works well for some class of operators (e.g., normal operators), but not some others. For equations preserving the Hamiltonian structure, the linearized operator around a solitary wave has essential spectrum on the imaginary axis, which corresponds to center manifold part that is hard to deal with. Another treatment is by the Lyaponov method, which is by Benjamin [2] and Bona [3], and later generalized to handle a class of Hamiltonian models by Weinstein [22] and Grillakis–Shatah–Strauss (GSS) [11]. They claim that knowing the information from the Lagrangian action energy allows one to determine the orbital stability and instability. The gain of their method is that instead of working with the original linearized operator, one just needs to study the spectrum of a rather transparent self-adjoint operator. The trade-off is that it is required to carefully weave the domain of the energy functional to balance between the complexity and solvability (due to loss of information from the original problem).

Besides the unidirectional models like KdV and CH, one can also allow transverse effect into modeling, leading to two-dimensional generalizations of the scalar models. Since the transverse perturbation is weak, it is natural to ask whether these models retain transverse stability, i.e., the unidirectional solitary waves remain stable under the two-dimensional flow. However, the answer to this question is much more involved. The first result is by Alexander–Pego–Sachs [1] on the Kadomtsev–Petviashvili (KP) equation

$$(u_t+uu_x+u_{xxx})_x-\sigma u_{yy}=0,$$

which is a two-dimensional version of the KdV equation. The coefficient σ takes values in $\{-1,1\}$ representing the strength of capillarity relative to the gravitational forces. The weak surface tension case corresponds to $\sigma=1$ and is referred to as the KP-I equation; and the strong surface tension leads to the so-called KP-II equation with $\sigma=-1$. In [1], the authors state that the KP-I model is linearly stable, while the KP-II model is linearly unstable. The transition from linear instability to nonlinear