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On a Rayleigh-Faber-Krahn Inequality for the Regional Fractional Laplacian

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Abstract. We study a Rayleigh-Faber-Krahn inequality for regional fractional Laplacian operators. In particular, we show that there exists a compactly supported nonnegative Sobolev function u_0 that attains the infimum (which will be a positive real number) of the set

$$\left\{\iint_{\{u>0\}\times\{u>0\}}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2\sigma}}\mathrm{d}x\mathrm{d}y : u\in \mathring{H}^{\sigma}(\mathbb{R}^n),\ \ \int_{\mathbb{R}^n}u^2=1,\ \ |\{u>0\}|\le 1\right\}.$$

Unlike the corresponding problem for the usual fractional Laplacian, where the domain of the integration is $\mathbb{R}^n\times\mathbb{R}^n$, symmetrization techniques may not apply here. Our approach is instead based on the direct method and new *a priori* diameter estimates. We also present several remaining open questions concerning the regularity and shape of the minimizers, and the form of the Euler-Lagrange equations.

AMS subject classifications: 35R11, 49Q10

Key words: Rayleigh-Faber-Krahn inequality, regional fractional Laplacian, first eigenvalue.

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1 Introduction

Let $n\geq 1$, $\sigma\in(0,1)$ (with the additional assumption that $\sigma<1/2$ if $n=1$), and $\Omega\subset\mathbb{R}^n$ be an open set. There are two natural fractional Sobolev norms which may be defined for $u\in C_c^\infty(\Omega)$:

$$I_{n,\sigma,\mathbb{R}^n}[u]:=\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{(u(x)-u(y))^2}{|x-y|^{n+2\sigma}}\mathrm{d}x\mathrm{d}y$$

and

$$I_{n,\sigma,\Omega}[u]:=\iint_{\Omega\times\Omega}\frac{(u(x)-u(y))^2}{|x-y|^{n+2\sigma}}\mathrm{d}x\mathrm{d}y.$$

Depending on the choices of n, σ and Ω, these two norms may or may not be equivalent. Even when they are equivalent (see Lemma 2.1), there are still subtle differences in how they depend on the domain Ω.

One significant difference is the behavior of their corresponding best Sobolev constants:

$$S_{n,\sigma}(\Omega):=\inf\left\{I_{n,\sigma,\Omega}[u]:u\in C_c^\infty(\Omega),\quad\int_\Omega|u|^{\frac{2n}{n-2\sigma}}\mathrm{d}x=1\right\}$$

and

$$\widetilde{S}_{n,\sigma}(\Omega):=\inf\left\{I_{n,\sigma,\mathbb{R}^n}[u]:u\in C_c^\infty(\Omega),\quad\int_\Omega|u|^{\frac{2n}{n-2\sigma}}\mathrm{d}x=1\right\}.$$

Clearly, $\widetilde{S}_{n,\sigma}(\Omega)\geq\widetilde{S}_{n,\sigma}(\mathbb{R}^n)$ and, in fact, using the dilation or translation invariance of $\widetilde{S}_{n,\sigma}(\mathbb{R}^n)$, it is not difficult to see that

$$\widetilde{S}_{n,\sigma}(\Omega)=\widetilde{S}_{n,\sigma}(\mathbb{R}^n)=S_{n,\sigma}(\mathbb{R}^n).$$

Moreover, a result of Lieb [15], classifies all minimizers for $\widetilde{S}_{n,\sigma}(\mathbb{R}^n)$ and shows that they do not vanish anywhere on $\mathbb{R}^n$. Therefore, the infimum $\widetilde{S}_{n,\sigma}(\Omega)$ is not attained unless $\Omega=\mathbb{R}^n$.

However, in [10], two of the authors with R. Frank discovered that the minimization problem for $S_{n,\sigma}(\Omega)$ behaves differently from $\widetilde{S}_{n,\sigma}(\Omega)$. Let us first recall some qualitative results about whether the constant $S_{n,\sigma}(\Omega)$ is positive or zero:

- For $n\geq 2$ and $\sigma>1/2$, one has $S_{n,\sigma}(\Omega)>0$ for any open set Ω. This follows from Dyda-Frank [8], which even shows that $\underline{S}_{n,\sigma}:=\inf_\Omega S_{n,\sigma}(\Omega)>0$.

- When $n\geq1$ and $\sigma<1/2$, one has $S_{n,\sigma}(\Omega)=0$ for any open set Ω of finite measure with sufficiently regular boundary; see Lemma 16 in [10].