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On the Cahn-Hilliard-Brinkman Equations in $\mathbb{R}^4$: Global Well-Posedness

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Abstract. We study the global well-posedness of large-data solutions to the Cauchy problem of the energy critical Cahn-Hilliard-Brinkman equations in $\mathbb{R}^4$. By developing delicate energy estimates, we show that for any given initial datum in $H^5(\mathbb{R}^4)$, there exists a unique global-in-time classical solution to the Cauchy problem. As a special consequence of the result, the global well-posedness of large-data solutions to the energy critical Cahn-Hilliard equation in $\mathbb{R}^4$ follows, which has not been established since the model was first developed over 60 years ago. The proof is constructed based on extensive applications of Gagliardo-Nirenberg type interpolation inequalities, which provides a unified approach for establishing the global well-posedness of large-data solutions to the energy critical Cahn-Hilliard and Cahn-Hilliard-Brinkman equations for spatial dimension up to four.

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1 Introduction

1.1 Background and object

The Cahn-Hilliard equation [7, 8]:

$$\partial_t\phi = M\Delta\left[-\varepsilon\Delta\phi + \varepsilon^{-1}\left(\phi^3 - \phi\right)\right] \tag{1.1}$$

is one of the fundamental models in mathematical physics, describing the evolutionary process of phase separation of binary fluids. Here, the unknown function ϕ denotes the relative difference of fluid concentrations, and $\mu \equiv -\varepsilon\Delta\phi + \varepsilon^{-1}(\phi^3 - \phi)$ stands for the chemical potential which can be derived from a coarse-grained study of the free energy of the fluid (c.f. [20]). The positive parameters M and ε model the mobility and diffusive interface thickness, respectively.

Besides phase separation, the Cahn-Hilliard equation also appears in modeling many other phenomena, including the evolution of two components of intergalactic material [45], dynamical interaction of two populations [10], modeling of bacterial film [25], and thin film problems [42, 44]. Since its initiation in the 1950's, the Cahn-Hilliard equation has been serving as a foundation for the mathematical modeling of phase separation. The success of the model is demonstrated through its capability of capturing the essential features of spinodal decomposition (anti-nucleation).

Because of its physical background and mathematical features, the qualitative and quantitative behaviors of the Cahn-Hilliard equation have been analyzed in the mathematics literature to a great extent. We refer the reader to [1, 2, 9, 14, 16, 28, 37–39, 41, 47] for analytical investigations, to [22, 23] for numerical simulations, and to [27, 29–34] for important progress recently made on the numerical analysis of the model.

Meanwhile, because phase separation appears in many fluid-related problems, of great interest to researchers in applied sciences is the coupling of the Cahn-Hilliard equation with fluid dynamics equations.

For example, the Cahn-Hilliard-Navier-Stokes (CHNS) system:

$$\partial_t\phi + \nabla\cdot(\mathbf{u}\phi) = M\Delta\mu, \tag{1.2a}$$

$$\mu = -\varepsilon\Delta\phi + \varepsilon^{-1}\left(\phi^3 - \phi\right), \tag{1.2b}$$

$$\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla\pi = \nu\Delta\mathbf{u} + \gamma\mu\nabla\phi, \tag{1.2c}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{1.2d}$$

and its variants have been utilized to study phase separation in general incompressible fluid flows (c.f. [4, 17, 21, 35]).