Negligible Obstructions and Turán Exponents

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Abstract. We show that for every rational number \( r \in (1,2) \) of the form \( 2 - a/b \), where \( a, b \in \mathbb{N}^+ \) satisfy

\[
[b/a]^2 \leq a \leq b/((b/a) + 1) + 1,
\]

there exists a graph \( F_r \) such that the Turán number \( \text{ex}(n,F_r) = \Theta(n^r) \). Our result in particular generates infinitely many new Turán exponents. As a byproduct, we formulate a framework that is taking shape in recent work on the Bukh–Conlon conjecture.

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1 Introduction

Given a family \( \mathcal{F} \) of graphs, the Turán number \( \text{ex}(n,\mathcal{F}) \) is defined to be the maximum number of edges in a graph on \( n \) vertices that contains no graph from the family \( \mathcal{F} \) as

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a subgraph. The classical Erdős–Stone–Simonovits theorem shows that arguably the most interesting problems about Turán numbers, known as the degenerate extremal graph problems, are to determine the order of magnitude of $ex(n, \mathcal{F})$ when $\mathcal{F}$ contains a bipartite graph. The following conjecture attributed to Erdős and Simonovits is central to Degenerate Extremal Graph Theory (see [16, Conjecture 1.6]).

**Conjecture 1.1** (Rational Exponents Conjecture). For every finite family $\mathcal{F}$ of graphs, if $\mathcal{F}$ contains a bipartite graph, then there exists a rational $r \in [1, 2)$ and a positive constant $c$ such that $ex(n, \mathcal{F}) = cn^r + o(n^r)$.

Recently Bukh and Conlon made a breakthrough on the inverse problem [16, Conjecture 2.37].

**Theorem 1.1** (Bukh and Conlon [3]). For every rational number $r \in (1, 2)$, there exists a finite family of graphs $\mathcal{F}_r$ such that $ex(n, \mathcal{F}_r) = \Theta(n^r)$.

Motivated by another outstanding problem of Erdős and Simonovits (see [10, Section III] and [11, Problem 8]), subsequent work has been focused on the following conjecture, which aims to narrow the family $\mathcal{F}_r$ in Theorem 1.1 down to a single graph.

**Conjecture 1.2** (Realizability of Rational Exponents). For every rational number $r \in (1, 2)$, there exists a bipartite graph $F_r$ such that $ex(n, F_r) = \Theta(n^r)$.

It is believed that the graph $F_r$ in Conjecture 1.2 could be taken from a specific yet rich family of graphs, for which we give the following definitions.

**Definition 1.1.** A rooted graph is a graph $F$ equipped with a subset $R(F)$ of vertices, which we refer to as roots. We define the $p$th power of $F$, denoted $F^p$, by taking the disjoint union of $p$ copies of $F$, and then identifying each root in $R(F)$, reducing multiple edges (if any) between the roots.

**Definition 1.2.** Given a rooted graph $F$, we define the density $\rho_F$ of $F$ to be $e(F)/(v(F) - |R(F)|)$, where $v(F)$ and $e(F)$ denote the number of vertices and respectively edges of $F$. We say that a rooted graph $F$ is balanced if $\rho_F > 1$, and for every subset $S$ of $V(F) \setminus R(F)$, the number of edges in $F$ with at least one endpoint in $S$ is at least $\rho_F |S|$.

Indeed the next result on Turán numbers, which follows immediately from [3, Lemma 1.2], establishes the lower bound in Conjecture 1.2 for some power of a balanced rooted tree.$^\dagger$

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$^\dagger$Erdős and Simonovits asked a much stronger question: for every rational number $r \in (1, 2)$, find a bipartite graph $F_r$ such that $ex(n, F_r) = cn^r + o(n^r)$ for some positive constant $c$.

$^\ddagger$A rooted tree is a rooted graph that is also a tree, not to be confused with a tree having a designated vertex.