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# Random Double Tensors Integrals

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**Abstract.** In this work, we try to build a theory for random double tensor integrals (DTI). We begin with the definition of DTI and discuss how randomness structure is built upon DTI. Then, the tail bound of the unitarily invariant norm for the random DTI is established and this bound can help us to derive tail bounds of the unitarily invariant norm for various types of two tensors means, e.g., arithmetic mean, geometric mean, harmonic mean, and general mean. By associating DTI with perturbation formula, i.e., a formula to relate the tensor-valued function difference with respect the difference of the function input tensors, the tail bounds of the unitarily invariant norm for the Lipschitz estimate of tensor-valued function with random tensors as arguments are derived for vanilla case and quasi-commutator case, respectively. We also establish the continuity property for random DTI in the sense of convergence in the random tensor mean, and we apply this continuity property to obtain the tail bound of the unitarily invariant norm for the derivative of the tensor-valued function.

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**Key words**: Einstein product, double tensor integrals (DTI), random DTI, tail bound, Lipschitz estimate, convergence in the random tensor mean, derivative of tensor-valued function.

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# 1 Introduction

In recent years, tensors have been applied to different applications in science and engineering [1–3]. However, most of these applications assume that systems modelled by tensors are deterministic and such assumption is not always true and practical in problems involving tensor formulations. In recent years, more research results have pioneered some theories about random tensors [4–6]. One important question in random tensors is about concentration behavior of random tensors. In [7], we extend Lapalace transform method and Lieb's concavity theorem from matrices to tensors, and apply these tools to generalize the classical bounds associated with the names Chernoff, Bennett, and Bernstein from the scalar to the tensor setting. In [8], this work extends previous work by considering the tail behavior of the top $k$-largest singular values of a function of the tensors summation, instead of the largest/smallest singular value of the tensors summation directly (identity function) explored in [7]. Majorization and antisymmetric tensor product tools are main techniques utilized to establish inequalities for unitarily norms of multivariate tensors. Random tensors summation form discussed in [7,8] is linear form, i.e., each summand of random tensors with degree one. In works [9,10], we extend the Hanson-Wright inequality for the maximum eigenvalue of the quadratic form of random Hermitian tensors under Einstein product. We separate the quadratic form of random tensors into diagonal summation and coupling (non-diagonal) summation parts. For the diagonal part, we can apply Bernstein inequality to bound the tail probability of the maximum eigenvalue [11] of the summation of independent random Hermitian tensors directly. For coupling summation part, we have to apply decoupling method first, i.e., decoupling inequality to bound expressions with dependent random Hermitian tensors with independent random Hermitian tensors, before applying Bernstein inequality again to bound the tail probability of the maximum eigenvalue of the coupling summation of independent random Hermitian tensors. Previous works are based on tensors with Einstein products. Since Kilmer et al. introduced the new multiplication method between two third-order tensors around 2008 and third-order tensors with such multiplication structure are also called as T-product tensors [12], T-product tensors have been applied to many fields in science and engineering, such as tensor computations [13–20], signal processing, image feature extraction, machine learning, computer vision, and the multi-view clustering problem, etc. The discussion about concentration behaviors based on T-product tensors can also be found in [21–23].

Inspired by operator mean theory (also called Kubo–Ando theory), we try to consider other operations besides + (arithmetic mean) among tensors [24]. The matrix mean for double operators can be expressed by Eq. (5:1:2) in [24], which has the same formation of double operator integral theory discussed in [25]. In this work,