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U-Eigenvalues' Inclusion Sets of Complex Tensors

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Abstract. In this paper, we study some inclusion sets of US-eigenvalues and U-eigenvalues based on quantum information. We give three inclusion sets theorems of US-eigenvalues and two inclusion sets theorems of U-eigenvalues. And we obtain the relationships among these inclusion sets. Some numerical examples are shown to illustrate the conclusions.

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Key words: Complex tensor, US-eigenvalue, U-eigenvalue, inclusion set.

1 Introduction

Let n be a positive integer and $[n]=\{1,2,\cdots,n\}$. Call

$$\mathcal{A}=(a_{i_1 i_2\cdots i_d}) \quad \text{for all} \;\; a_{i_1 i_2\cdots i_d}\in\mathbb{C}, \quad i_k\in[n_k], \quad k\in[d],$$

a d-order $(n_1\times n_2\times\cdots\times n_d)$-dimensional complex tensor. When $n_1=n_2=\cdots=n_d=n$, $\mathcal{A}$ is a d-order n-dimensional complex tensor. In particular, when $d=1$ and $d=2$, they are vector and matrix, respectively. Let $\mathbb{C}^{n_1\times n_2\times\cdots\times n_d}$ be the set of d-order $(n_1\times n_2\times\cdots\times n_d)$-dimensional tensors over $\mathbb{C}$.

In 2014, Ni et al. [1] proposed definitions of U-eigenvalues and US-eigenvalues based on quantum information, i.e., converting the geometric measure of the entanglement [2–4] problem to an algebraic equation system problem. Using an iterative

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algorithm, Che et al. [14] computed the U- and US-eigenpairs of complex tensors in 2017. In 2018, Che et al. [15] studied the geometric measures of entanglement in multipartite pure states via complex-valued neural networks. Due to the complexity of tensor operations, it is troublesome to computing the U- and US-eigenvalues of complex tensors. Sometimes, we only need to know the range of them. Therefore, the inclusion sets of U- and US- eigenvalues are given in this paper.

For $\mathcal{A},\mathcal{B}\in\mathbb{C}^{n_1\times n_2\times\cdots\times n_d}$, the inner product and norm are

$$\langle\mathcal{A},\mathcal{B}\rangle=\sum_{i_1,i_2,\cdots,i_d=1}^{n_1,n_2,\cdots,n_d}(\mathcal{A}^*)_{i_1i_2\cdots i_d}(\mathcal{B})_{i_1i_2\cdots i_d},$$
$$\|\mathcal{A}\|=\sqrt{\langle\mathcal{A},\mathcal{A}\rangle},$$

where $(\mathcal{A}^*)_{i_1i_2\cdots i_d}$ denotes the complex conjugate of $(\mathcal{A})_{i_1i_2\cdots i_d}$. A rank-one tensor is defined as $\otimes_{i=1}^d\boldsymbol{x}^{(i)}\in\mathbb{C}^{n_1\times n_2\times\cdots\times n_d}$, where $\boldsymbol{x}^{(i)}\in\mathbb{C}^{n_i},i\in[d]$. By tensor product,

$$\mathcal{A}^*\left(\boldsymbol{x}^{(1)},\cdots,\boldsymbol{x}^{(k-1)},I_{n_k},\boldsymbol{x}^{(k+1)},\cdots,\boldsymbol{x}^{(d)}\right),$$
$$\mathcal{A}\left(\boldsymbol{x}^{(1)*},\cdots,\boldsymbol{x}^{(k-1)*},I_{n_k},\boldsymbol{x}^{(k+1)*},\cdots,\boldsymbol{x}^{(d)*}\right),$$

for vectors $\boldsymbol{x}^{(i)}\in\mathbb{C}^{n_i}\,(i\in[d])$ denote vectors in $\mathbb{C}^{n_k}$, whose pth components are

$$\begin{aligned}&\left(\mathcal{A}^*\left(\boldsymbol{x}^{(1)},\cdots,\boldsymbol{x}^{(k-1)},I_{n_k},\boldsymbol{x}^{(k+1)},\cdots,\boldsymbol{x}^{(d)}\right)\right)_p\\&=\sum_{i_1,\cdots,i_{k-1},i_{k+1},\cdots,i_d=1}^{n_1,\cdots,n_{k-1},n_{k+1},\cdots,n_d}(\mathcal{A}^*)_{i_1\cdots i_{k-1}pi_{k+1}\cdots i_d}x_{i_1}^{(1)}\cdots x_{i_{k-1}}^{(k-1)}x_{i_{k+1}}^{(k+1)}\cdots x_{i_d}^{(d)},\end{aligned}\tag{1.1a}$$
$$\begin{aligned}&\left(\mathcal{A}\left(\mathbf{x}^{(1)*},\cdots,\boldsymbol{x}^{(k-1)*},I_{n_k},\boldsymbol{x}^{(k+1)*},\cdots,\boldsymbol{x}^{(d)*}\right)\right)_p\\&=\sum_{i_1,\cdots,i_{k-1},i_{k+1},\cdots,i_d=1}^{n_1,\cdots,n_{k-1},n_{k+1},\cdots,n_d}(\mathcal{A})_{i_1\cdots i_{k-1}pi_{k+1}\cdots i_d}x_{i_1}^{(1)*}\cdots x_{i_{k-1}}^{(k-1)*}x_{i_{k+1}}^{(k+1)*}\cdots x_{i_d}^{(d)*},\end{aligned}\tag{1.1b}$$

where I_{n_k} is a $n_k\times n_k$ identity matrix, $p\in[n_k]$, $k\in[d]$.

A tensor $\mathcal{S}=(s_{i_1i_2\cdots i_d})\in\mathbb{C}^{n\times n\times\cdots\times n}$ is called complex symmetric if its entries $s_{i_1i_2\cdots i_d}$ are invariant under any permutation of their indices. Let $\boldsymbol{x}\in\mathbb{C}^n$, similarly,

$$\mathcal{S}^*(I_n,\boldsymbol{x},\cdots,\boldsymbol{x})\in\mathbb{C}^n,$$
$$\mathcal{S}(I_n,\boldsymbol{x}^*,\cdots,\boldsymbol{x}^*)\in\mathbb{C}^n,$$