Ann. Appl. Math.
doi: 10.4208/aam.OA-2023-0020

Vol. **39**, No. 4, pp. 544-569
November 2023

Specht Triangle Approximation of Large Bending Isometries

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Received 2 June 2023; Accepted (in revised version) 8 October 2023

Dedicated to the memory of Professor Zhongci Shi

Abstract. We propose a Specht triangle discretization for a geometrically nonlinear Kirchhoff plate model with large bending isometry. A combination of an adaptive time-stepping gradient flow and a Newton's method is employed to solve the ensuing nonlinear minimization problem. $\Gamma-$convergence of the Specht triangle discretization and the unconditional stability of the gradient flow algorithm are proved. We present several numerical examples to demonstrate that our approach significantly enhances both the computational efficiency and accuracy.

AMS subject classifications: 65N12, 65N30, 74K20

Key words: Specht triangle, plate bending, isometry constraint, adaptive time-stepping gradient flow.

1 Introduction

The geometrically nonlinear Kirchhoff plate models have drawn great attention recently because they capture the critical feature of large bending deformations of

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thin plates in modern nanotechnological applications [9, 19, 21, 26, 29]. The dimensional reduced nonlinear plate model has been proposed by Kirchhoff in 1850 [20], which is based on a curvature functional subject to a pointwise isometry constraint. Friesecke and Müller [17] have derived this model from three dimensional hyperelasticity via $\Gamma-$convergence. Since then, extensive studies have been carried out from the perspective of modeling and numerics, such as the single layer problem [4, 13], the bilayer problem [1, 8, 12], the thermally actuated bilayer problem [7], just to mention a few. The above models all involve minimizing an energy functional with the isometry constraint. One of the numerical difficulties is the non-convexity of the energy functional caused by the isometry constraint [6].

In this work we focus on the single layer model, and find a deformation $y:\Omega\to\mathbb{R}^3$ of a bounded Lipschitz domain $\Omega\subset\mathbb{R}^2$ by minimizing

$$E[y]=\frac{1}{2}\int_\Omega |H|^2\mathrm{d}x-\int_\Omega f(x)\cdot y(x)\mathrm{d}x \tag{1.1}$$

subject to the isometry constraint

$$(\nabla y)^\top(\nabla y)=\mathrm{Id}_2 \quad \text{a.e. in } \ \Omega, \tag{1.2}$$

and Dirichlet boundary condition

$$y(x)=g,\quad \nabla y=\Phi,\quad \Phi^\top\Phi=\mathrm{Id}_2 \quad \text{on } \ \Gamma_D,$$

where $H\in L^2(\Omega;\mathbb{R}^{2\times 2})$ is the second fundamental form of the parametrized surface given by

$$H_{ij}=n\cdot\partial_i\partial_j y=(\partial_1 y\times\partial_2 y)\cdot\partial_i\partial_j y,\quad i,j=1,2,$$

and Id_2 is the identity matrix of order 2. Γ_D is part of the boundary of D. We assume that

$$g\in H^2(\Omega;\mathbb{R}^3),\quad \Phi\in H^2(\Omega;\mathbb{R}^{3\times 2}),\quad f\in L^2(\Omega,\mathbb{R}^3). \tag{1.3}$$

The numerical approximation of this problem consists of two parts: discretization and minimization. A proper finite element discretization needs to take into account both high order differential operator of (1.1) and the pointwise isometry constraint. In [4, 8], the authors employed a discrete Kirchhoff triangle (DKT), which has been developed in [10] for the linear bending problem. DKT element possess the degrees of freedom of gradient at the nodes, which is convenient for imposing the constraint (1.2). The implementation of DKT element is based on the construction of a discrete gradient operator that maps the gradient to another space. Although this operator makes the proof of $\Gamma-$convergence of the discrete energy easier, while its numerical realization is rather complicate. Moreover, DKT