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On a Hybrid Method for Inverse Transmission Eigenvalue Problems

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Abstract. In this paper, we are concerned with the inverse transmission eigenvalue problem to recover the shape as well as the constant refractive index of a penetrable medium scatterer. The linear sampling method is employed to determine the transmission eigenvalues within a certain wavenumber interval based on far-field measurements. Based on a prior information given by the linear sampling method, the neural network approach is proposed for the reconstruction of the unknown scatterer. We divide the wavenumber intervals into several subintervals, ensuring that each transmission eigenvalue is located in its corresponding subinterval. In each such subinterval, the wavenumber that yields the maximum value of the indicator functional will be included in the input set during the generation of the training data. This technique for data generation effectively ensures the consistent dimensions of model input. The refractive index and shape are taken as the output of the network. Due to the fact that transmission eigenvalues considered in our method are relatively small, certain super-resolution effects can also be generated. Numerical experiments are presented to verify the effectiveness and promising features of the proposed method in two and three dimensions.

AMS subject classifications: 35P25, 35R30, 35P15

Key words: Inverse transmission eigenvalue problem, linear sampling method, neural network, super-resolution.

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1 Introduction

In this paper, we are mainly concerned with the interior transmission eigenvalue problem and its applications to the inverse scattering problem. This problem has received considerable interest in the literature in recent years [10, 14]. In what follows, we first present the mathematical formulation of our study.

Let $D_0\subset\mathbb{R}^n$ $(n=2,3)$ be a bounded and simply connected domain with a smooth boundary Γ_0. The incident field is given by the time-harmonic plane wave in the form

$$u^i(x):=e^{\mathrm{i}kx\cdot d}, \tag{1.1}$$

where $k\in\mathbb{R}_+$ is the wavenumber, $d\in\mathbb{S}^{n-1}:=\{x\in\mathbb{R}^n;|x|=1\}$ is the incident direction and $\mathrm{i}:=\sqrt{-1}$. The exterior scattering problem is to find the total field $u=u^s(x,d,D_0)+u^i(x)$ and the transmitted field u_0 satisfying

$$\begin{cases} \Delta u+k^2u=0 & \text{in } \mathbb{R}^n\backslash\overline{D_0}, \\ \Delta u_0+k^2n_0u_0=0 & \text{in } D_0, \\ u-u_0=0, \quad \dfrac{\partial u}{\partial\nu}-\dfrac{\partial u_0}{\partial\nu}=0 & \text{on } \Gamma_0, \\ \lim\limits_{r\to\infty} r^{\frac{n-1}{2}}\left(\dfrac{\partial u^s(x,d,D_0)}{\partial r}-\mathrm{i}ku^s(x,d,D_0)\right)=0, & r=|x|, \end{cases} \tag{1.2}$$

where $n_0\in L^\infty(D_0)$ is the positive refractive index and ν is the unit outward normal. There exists a unique solution $u_0\chi_{D_0}+u\chi_{\mathbb{R}^n\backslash\overline{D}_0}\in H^1_{loc}(\mathbb{R}^n)$ to the problem (1.2) (cf. [21]). The associated interior transmission eigenvalue problem to (1.2) for nontrivial $(v,w)\in L^2(D_0)\times L^2(D_0)$ can be described by

$$\begin{cases} \Delta v+k^2v=0 & \text{in } D_0, \\ \Delta w+k^2n_0w=0 & \text{in } D_0, \\ v-w=0, \quad \dfrac{\partial v}{\partial\nu}-\dfrac{\partial w}{\partial\nu}=0 & \text{on } \Gamma_0. \end{cases} \tag{1.3}$$

The wavenumber $k\in\mathbb{R}_+$ is called an interior transmission eigenvalues with the associated transmission eigenfunctions v, w. In order to generate certain super-resolution effects, a relatively large refractive index should be chosen to obtain the relatively small transmission eigenvalues. For practical considerations, we assume that the refractive index of the medium scatterer D_0 is relatively small. In order to produce the super-resolution effect, we introduce a thin coating D_1 on the domain D_0 (see Fig. 1), $\Gamma_1:=\partial D_1$. The refractive index n_1 in D_1 is far larger than n_0 in D_0, i.e., $n_0\ll n_1$. For the subsequent discussion, we regard the domains D_1 and D_0 as the