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Numerical Study of Semidiscrete Penalty Approach for Stabilizing Boussinesq System with Localized Feedback Control

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Abstract. We investigate the numerical approximation for stabilizing the semidiscrete linearized Boussinesq system around an unstable stationary state. Stabilization is attained through internal feedback controls applied to the velocity and temperature equations, localized within an arbitrary open subset. This study follows the framework presented in [14], considering the continuous linearized Boussinesq system. The primary objective is to explore the penalization-based approximation of the free divergence condition in the semidiscrete case and provide a numerical validation of these results in a two-dimensional setting.

AMS subject classifications: 49N10, 93B05

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1 Introduction

The optimal control of the Boussinesq system and its linearization around a stationary state are topics of significant interest across various application fields, including the design and operation of energy-efficient buildings (see, for instance, [4,6,17,21]).

In the recent paper [14], the authors examine the penalization-based approximation of the free divergence condition for the Linear Quadratic Regulator (LQR)

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optimal control problem associated with the continuous linearized Boussinesq system around a stationary state, considering an infinite time horizon. The primary theoretical motivation for such an approximation is to represent the system as a well-posed control system in the sense of Salomon-Weiss (as described, for instance, in Curtain and Weiss [9]). From a numerical standpoint, the secondary advantage is the avoidance of projection methods for handling the free divergence condition. It's worth noting that efficient projection-based numerical methods have been developed in recent literature, as seen in [3, 12, 18] and more recently [2].

This article focuses on the numerical stabilization of the linearized Boussinesq system through localized feedback controls. Specifically, we explore the penalty approach for the semidiscrete linearized Boussinesq system.

Let's begin by introducing the model under consideration. Suppose Ω is a smooth domain contained in $\mathbb{R}^d$ where $d=2,3$. Let $\mathcal{O}$ be a nonempty open subset within Ω, $\Gamma:=\partial\Omega$, and n represents the outer unit normal vector. The incompressible Boussinesq system with Neumann boundary conditions writes as follows:

$$\begin{cases} \partial_t v - \text{div}\ \sigma(v,p) + (v\cdot\nabla)v = ye_d + f & \text{in} \ \ (0,\infty)\times\Omega, \\ \partial_t y - \alpha\Delta y + v\cdot\nabla y = g & \text{in} \ \ (0,\infty)\times\Omega, \\ \text{div}\ v = 0 & \text{in} \ \ (0,\infty)\times\Omega, \\ \sigma(v,p)n = t, \quad \partial_n y = k & \text{on} \ \ (0,\infty)\times\Gamma, \\ v(0,\cdot) = v_0, \quad y(0,\cdot) = y_0 & \text{in} \ \ \Omega. \end{cases} \tag{1.1}$$

In (1.1), v denotes the fluid velocity, p is the fluid pressure, y is the temperature of the fluid, $\sigma(v,p)=\nu((\nabla v)+(\nabla v)^{\text{tr}})-pI$ is the Cauchy stress tensor, $\nu>0$ is the kinematic viscosity of the fluid, $\alpha>0$ is the heat conductivity of the fluid, e_d is the last vector of the canonical basis of $\mathbb{R}^d$. The terms $f:\Omega\to\mathbb{R}^d$, $g:\Omega\to\mathbb{R}$ describe respectively the influence of internal field forces and heat sources. The boundary term $t:\Gamma\to\mathbb{R}^d$ is a traction boundary condition which is an example of non-reflecting outlet boundary, it is known to be efficient for low Reynolds number, see [15]. The Neumann boundary condition $k:\Gamma\to\mathbb{R}$ prescribes the heat flux.

We assume that $(v_s,p_s,y_s)\in W^{1,\infty}(\Omega;\mathbb{R})^{d+2}$ is a real-valued solution to the stationary Boussinesq system

$$\begin{cases} -\text{div}\ \sigma(v_s,p_s) + (v_s\cdot\nabla)v_s = y_s e_d + f & \text{in} \ \ \Omega, \\ -\alpha\Delta y + v_s\cdot\nabla y_s = g & \text{in} \ \ \Omega, \\ \text{div}\ v_s = 0 & \text{in} \ \ \Omega, \\ \sigma(v,p)n = h, \quad \partial_n y = k & \text{on} \ \ \Gamma. \end{cases} \tag{1.2}$$