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EXISTENCE AND UNIQUENESS OF GLOBAL SOLUTIONS OF THE MODIFIED KS-CGL EQUATIONS FOR FLAMES GOVERNED BY A SEQUENTIAL REACTION*

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Abstract

In this paper, we are concerned with the existence and uniqueness of global solutions of the modified KS-CGL equations for flames governed by a sequential reaction, where the term $|P|^2P$ is replaced with the generalized form $|P|^{2\sigma}P$, see [18]. The main novelty compared with [18] in this paper is to control the norms of the first order of the solutions and extend the global well-posedness to three dimensional space.

Keywords existence and uniqueness; modified KS-CGL; global solutions

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1 Introduction

This paper is devoted to the existence and uniqueness of global solutions of the following coupled modified Kuramoto-Sivashinsky-complex Ginzburg-Landau (GKS-CGL) equations for flames

$$\partial_t P = \xi P + (1 + i\mu)\Delta P - (1 + i\nu)|P|^{2\sigma} P - \nabla P \nabla Q - r_1 P \Delta Q - g r_2 P \Delta^2 Q, \quad (1.1)$$

$$\partial_t Q = -\Delta Q - g\Delta^2 Q + \delta\Delta^3 Q - \frac{1}{2}|\nabla Q|^2 - \eta|P|^2, \quad (1.2)$$

with the periodic initial conditions

$$P(x + Le_i, t) = P(x, t), \quad Q(x + Le_i, t) = Q(x, t), \quad x \in \Omega, \ t \geq 0, \quad (1.3)$$

$$P(x, 0) = P_0(x), \quad Q(x, 0) = Q_0(x), \quad x \in \Omega, \quad (1.4)$$

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where Ω is a box with length L denoted T^n $(n = 1, 2, 3)$. The complex function $P(x,t)$ denotes the rescaled amplitude of the flame oscillations, and the real function $Q(x,t)$ is the deformation of the first front. The Landau coefficients μ, ν and the coupling coefficient $\eta > 0$ are real, while r_1 and r_2 are complex parameters of the form $r_1 = r_{1r} + ir_{1i}$, $r_2 = r_{2r} + ir_{2i}$, respectively. The coefficient $g > 0$ is proportional to the supercriticality of the oscillatory mode, $\delta > 0$ is a constant, $L > 0$ is the period and e_i is the standard coordinate vector, and the coefficient $\xi = \pm 1$. The parameter σ, μ, ν satisfy

$$\text{(A1)} \quad 1 < \sigma < \frac{1}{\sqrt{1 + (1 + \frac{\mu - \nu}{2}^2)} - 1},$$

$$\text{(A2)} \quad \sigma \leq \frac{\sqrt{1+\mu^2}}{4\sqrt{1+\mu^2} - 4}.$$

For $\delta = 0$, the coupled GKS-CGL equations (1.1) and (1.2) are reduced to the classical KS-CGL equations [1], which describe the nonlinear interaction between the monotonic and oscillatory modes of instability of the two uniformly propagating flame fronts in a sequential reaction. Specifically, they describe both the long-wave evolution of the oscillatory mode near the oscillatory instability threshold and the evolution of the monotonic mode. For the background of the uniformly propagating premixed flame fronts and the derivation of the KS-CGL model, one refer to [1-4] for details. If there exist no coupling with the monotonic model, then equation (1.1) is the well-known CGL equation that describes the weakly nonlinear evolution of a long-scaled instability [5]. For $\delta = 0$ and the coupled coefficient $\eta = 0$ in equation (1.2), equation (1.2) reduces to the well known KS equation [6], which governs the flame front's spatio-temporal evolution and produces monotonic instability. It's seen that the coupled GKS-CGL equations (1.1) and (1.2) can better describe the dynamical behavior for flames governed by a sequential reaction, since they generalize the KS equations, the CGL equations, and the KS-CGL equations.

So far, the mathematical analysis and physical study about the CGL equation and KS equation have been done by many researchers. For example, the existence of global solutions and the attractor for the CGL equation were studied in [7-11]. For some other results, see [12-15] and reference therein. However, little progress has been made for the coupled KS-CGL equations which are derived to describe the nonlinear evolution for flames by A.A. Golovin, et. al. [1], who studied the traveling waves of the coupled equations numerically and the spiral waves in [16], where new types of instabilities are exhibited. Meanwhile, there are few works to consider mathematical analytical properties of the KS-CGL equations and the generalized KS-CGL equations, even the existence and uniqueness of the solutions.

In [17], the Littlewood-Paley theory is used to obtain the local solution and global solution with small initial conditions for the coupled KS-Burgers which are derived by [1], while in [18] an additional sixth order term is added to control the nonlinear term estimate through which one can get the global smooth solution of the generalized GKS-CGL equation. In this paper, via delicate a prior estimates and the Galerkin method, we consider the global smooth solution of GKS-CGL equations where the third term is replaced by $(1 + i\nu)|P|^{2\sigma}P$, that is, we study system (1.1)-(1.4) and extend the global well-posedness to the three dimensional space.

The rest of this paper is organized as follows. In Section 2, we briefly give some notations and preliminaries. In Section 3, local solutions are constructed by the contraction mapping theorem. A prior estimates for the solutions of the periodic initial value problem (1.1)-(1.4) are obtained in Section 4. In Section 5, we deduce from the so-called continuity method that the existence and uniqueness of the global solutions of the periodic initial value problem (1.1)-(1.4).

2 Notaions and Preliminaries

For convenience, we will recall some notations and preliminaries which will be used in the sequel.

Let L^k_{per} and H^k_{per} $(k = 1, 2, \cdots)$ denote the Sobolev spaces of L-periodic and complex-valued functions respectively endowed with norms

$$\|u\|_p = \left(\int_\Omega |u|^p \mathrm{d}x\right)^{1/p}, \quad \|u\|_{H^k} = \left(\sum_{|a|\le k} \|D^\alpha u(x)\|\right)^2,$$

here we write $\|u\| = \|u\|_{L^2} = \sqrt{(u,u)}$, where the inner product $(\cdot,\cdot)$ is defined by $(u,v) = \int_\Omega u(x)\overline{v(x)}\mathrm{d}x$ and $\overline{v}$ denotes the complex conjugate of v.

Now, we give some useful inequalities.

Lemma 2.1[19](Young's inequality with ε) *Let $a > 0$, $b > 0$, $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$ab \le \varepsilon a^p + C(\varepsilon) b^q,$$

for $C(\varepsilon) = (\varepsilon p)^{-q/p} q^{-1}$.

Lemma 2.2[20](Gagliardo-Nirenberg inequality) *Let Ω be a bounded domain with $\partial\Omega$ in C^m, and u be any function in $W^{m,r} \cap L^q(\Omega)$, $1 \le q, r \le \infty$. For any integer j, $0 \le j < m$ and for any number a in the interval $j/m \le a \le 1$, set*

$$\frac{1}{p} = \frac{j}{n} + a\Big(\frac{1}{r} - \frac{m}{n}\Big) + (1-a)\frac{1}{q}.$$

If $m-j-n/r$ is not nonnegative integer, then

$$\|D^j u\|_p \le C\|u\|_{W^{m,r}}^a \|u\|_{L^q}^{1-a}. \tag{2.1}$$

If $m-j-n/r$ is a nonnegative integer, then (2.1) *holds for $a=j/m$. The constant C depends only on Ω, r, q, j and a.*

In the sequel, we will use the following inequalities which are the specific cases of the Gagliardo-Nirenberg inequality:

$$\|D^j u\|_\infty \le C\|u\|_{H^m}^a \|u\|^{1-a}, \quad ma = j + n/2, \tag{2.2}$$

$$\|D^j u\|_2 \le C\|u\|_{H^m}^a \|u\|^{1-a}, \quad ma = j, \tag{2.3}$$

$$\|D^j u\|_4 \le C\|u\|_{H^m}^a \|u\|^{1-a}, \quad ma = j + n/4. \tag{2.4}$$

3 Local Solution

In this section, we will use the contraction mapping principle to prove the local solution of system (1.1)-(1.4).

For convenience, we rewrite the system as an abstract form

$$\frac{\mathrm{d}P}{\mathrm{d}t} + AP = M(P,Q), \tag{3.1}$$

$$\frac{\mathrm{d}Q}{\mathrm{d}t} + BQ = N(P,Q), \tag{3.2}$$

where $A=-(1+i\mu)\Delta$ with domain $D(A)=H^1(\Omega)$, $M(p,Q)=\xi P-(1+i\nu)|P|^{2\sigma}P-\nabla P\nabla Q - r_1 P\Delta Q - gr_2 P\Delta^2 Q$; $B=-\delta\Delta^3$ with domain $D(A)=H^5(\Omega)$, $N(p,Q)=-\Delta Q - g\Delta^2 Q - \frac{1}{2}|\nabla Q|^2 - \eta|P|^2$. Then operators $-A$ and $-B$ generate uniformly bounded analytic semigroup $S_1(t)=\mathrm{e}^{-At}$ and $S_2(t)=\mathrm{e}^{-Bt}$ for $t\ge 0$, respectively. Therefore, we deduce from (3.1), (3.2) that

$$P = S_1(t)P_0 + \int_0^t S_1(t-s)M(P,Q)\mathrm{d}s, \tag{3.3}$$

$$Q = S_2(t)Q_0 + \int_0^t S_2(t-s)N(P,Q)\mathrm{d}s. \tag{3.4}$$

Then, define a mapping

$$\Re : (P,Q) \to (\overline{P},\overline{Q}), \tag{3.5}$$

where

$$\overline{P} = S_1(t)P_0 + \int_0^t S_1(t-s)M(P,Q)\mathrm{d}s, \tag{3.6}$$

$$\overline{Q} = S_2(t)Q_0 + \int_0^t S_2(t-s)N(P,Q)\mathrm{d}s, \tag{3.7}$$

and define a normed linear space

$$\Im=\{(P,Q)\in C([0,T];H^1(\Omega))\times C([0,T];H^5(\Omega))|\sup_{[0,T]}\|P\|_{H^1}\le C\|P_0\|_{H^1}+1\equiv R,$$
$$\sup_{[0,T]}\|Q\|_{H^5}\le C\|Q_0\|_{H^1}+1\equiv R\},$$

therefore,

$$\begin{aligned}\|\overline{P}\|_{H^1}&\le C\|P_0\|_{H^1}+\int_0^t\|A^{1/2}\mathrm{e}^{-A(t-\tau)}M(P,Q)\|_{L^2}\mathrm{d}\tau\\&\le C\|P_0\|_{H^1}+T^{1/2}\|M(P,Q)\|_{L^2},\end{aligned}\tag{3.8}$$

$$\begin{aligned}&\|M(P,Q)\|_{L^2}\\&=\|\xi P-(1+i\nu)|P|^{2\sigma}P-\nabla P\nabla Q-r_1P\Delta Q-gr_2P\Delta^2Q\|_{L^2}\\&\le C\|P\|_{L^2}+R^{2\sigma}\|P\|_{H^1}+C\|\nabla P\|_{L^2}\|Q\|_{H^4}+C\|P\|_{L^2}\|Q\|_{H^4}+\|P\|_{L^4}\|\Delta^2Q\|_{L^4}\\&\le C\|P\|_{L^2}+R^{2\sigma}\|P\|_{H^1}+C\|\nabla P\|_{L^2}\|Q\|_{H^4}+C\|P\|_{L^2}\|Q\|_{H^4}+\|P\|_{H^1}\|Q\|_{H^5},\end{aligned}$$

here the Sobolev imbedding theorem $H^1\hookrightarrow L^{2(2\sigma+1)}$ is used, then we obtain

$$\|\overline{P}\|_{H^1}\le C\|P_0\|_{H^1}+CT^{1/2}(1+R^{2\sigma}+R)R.\tag{3.9}$$

Therefore, when T is small, $\Re$ maps $\Im$ into itself. Similarly, we get

$$\begin{aligned}\|\overline{Q}\|_{H^5}&\le C\|Q_0\|_{H^5}+\int_0^t\|A^{5/6}\mathrm{e}^{-B(t-\tau)}N(P,Q)\|_{L^2}\mathrm{d}\tau\\&\le C\|Q_0\|_{H^5}+T^{1/6}\|N(P,Q)\|_{L^2},\end{aligned}\tag{3.10}$$

$$\begin{aligned}\|N(P,Q)\|_{L^2}&=\|\Delta Q+g\Delta^2Q-\frac{1}{2}|\nabla Q|^2-\eta|P|^2\|_{L^2}\\&\le C\|\Delta Q\|_{L^2}+C\|\Delta^2Q\|_{L^2}+C\|\nabla Q\|_{L^\infty}\|\nabla Q\|_{L^2}+C\|P\|_{L^4}^2,\end{aligned}$$

then we have

$$\|\overline{Q}\|_{H^5}\le C\|Q_0\|_{H^5}+CT^{1/6}(1+R)R.\tag{3.11}$$

Therefore, when T is small, $\Re$ maps $\Im$ into itself.

Now, we prove $\Re$ is a contractive mapping on $\Im$, since

$$\|\Re(P_1)-\Re(P_2)\|_{H^1}\le\int_0^t\|A^{1/2}\mathrm{e}^{-A(t-\tau)}M(P_1-P_2,Q_1-Q_2)\|_{L^2}\mathrm{d}\tau,\tag{3.12}$$

$$\begin{aligned}\|M(P_1-P_2,Q_1-Q_2)\|_{L^2}\le\ &\|\xi(P_1-P_2)-(1+i\nu)(|P_1|^{2\sigma}P_1-|P_2|^{2\sigma}P_2)\\&-(\nabla P_1\nabla Q_1-\nabla P_2\nabla Q_2)-r_1(P_1\Delta Q_1-P_2\Delta Q_2)\\&-gr_2(P_1\Delta^2Q_1-P_2\Delta^2Q_2)\|_{L^2},\end{aligned}\tag{3.13}$$

where

$$\|(1+i\nu)(|P_1|^{2\sigma}P_1-|P_2|^{2\sigma}P_2)\|_{L^2}\le CR^{2\sigma}\|P_1-P_2\|_{L^2}\le CR^{2\sigma}\|P_1-P_2\|_{H^1},$$

$$\begin{aligned}\|\nabla P_1\nabla Q_1-\nabla P_2\nabla Q_2\|_{L^2}&=\|\nabla P_1\nabla Q_1-\nabla P_1\nabla Q_2+\nabla P_1\nabla Q_2-\nabla P_2\nabla Q_2\|_{L^2}\\&\le\|\nabla Q_1-\nabla Q_2\|_{L^\infty}\|\nabla P_1\|_{L^2}+\|\nabla P_1-\nabla P_2\|_{L^2}\|\nabla Q_2\|_{L^\infty}\\&\le CR(\|P_1-P_2\|_{H^1}+\|Q_1-Q_2\|_{H^5}),\\\|r_1(P_1\Delta Q_1-P_2\Delta Q_2)\|_{L^2}&=\|r_1(P_1\Delta Q_1-P_2\Delta Q_1+P_2\Delta Q_1-P_2\Delta Q_2)\|_{L^2}\\&\le C\|P_1-P_2\|_{L^2}\|\Delta Q_1\|_{L^\infty}+C\|\Delta Q_1-\Delta Q_2\|_{L^\infty}\|\nabla P_2\|_{L^2}\\&\le CR(\|P_1-P_2\|_{H^1}+\|Q_1-Q_2\|_{H^5}),\end{aligned}$$

$$\begin{aligned}\|gr_2(P_1\Delta^2Q_1-P_2\Delta^2Q_2)\|_{L^2}&=\|gr_2(P_1\Delta^2Q_1-P_2\Delta^2Q_1+P_2\Delta^2Q_1-P_2\Delta^2Q_2)\|_{L^2}\\&\le C\|P_1-P_2\|_{L^4}\|\Delta Q_1\|_{L^4}+C\|\Delta^2Q_1-\Delta^2Q_2\|_{L^4}\|P_2\|_{L^4}\\&\le C\|P_1-P_2\|_{H^1}\|Q_1\|_{H^5}+C\|Q_1-Q_2\|_{H^5}\|P_2\|_{H^1}\\&\le CR(\|P_1-P_2\|_{H^1}+\|Q_1-Q_2\|_{H^5}),\end{aligned}$$

then we have

$$\|\Re(P_1)-\Re(P_2)\|_{H^1}\le CT^{1/2}[(R+R^{2\sigma})\|P_1-P_2\|_{H^1}+R\|Q_1-Q_2\|_{H^5}]. \tag{3.14}$$

Similarly, we get

$$\|\Re(Q_1)-\Re(Q_2)\|_{H^5}\le CT^{1/2}R(\|P_1-P_2\|_{H^1}+\|Q_1-Q_2\|_{H^5}). \tag{3.15}$$

Adding the above two inequalities, we finally deduce

$$\begin{aligned}&\|\Re(P_1)-\Re(P_2)\|_{H^1}+\|\Re(Q_1)-\Re(Q_2)\|_{H^5}\\&\le CT^{1/2}[(R+R^{2\sigma})\|P_1-P_2\|_{H^1}+R\|Q_1-Q_2\|_{H^5}].\end{aligned}\tag{3.16}$$

By taking T so small that $CT^{1/2}(R+R^{2\sigma})\le\frac{1}{2}$, then $\Re$ is a contraction on $\Im$. We deduce from the contraction mapping principle that there exists a fixed point of $\Re$ on $\Im$, that is, there exists a unique local solution of system (1.1)-(1.4) such that

$$(P,Q)\in C([0,T];H^1(\Omega))\times C([0,T];H^5(\Omega)), \tag{3.17}$$

where T depends on $\|P_0\|_{H^1(\Omega)}$ and $\|Q_0\|_{H^5(\Omega)}$.

4 A Priori Estimates

In this section, we will derive the a priori estimates for the solutions of problem (1.1)-(1.4). Firstly, we have:

Lemma 4.1 *Let $P_0(x) \in L^2_{per}(\Omega)$, $Q_0(x) \in H^1_{per}(\Omega)$ and suppose that $\sigma > 1$ and Ω is a bounded domain with $\partial\Omega$ in C^m. Then*

$$\|P\|^2 \le \mathrm{e}^{Ct}(\|P_0\|^2 + \|\nabla Q_0\|^2 + Ct), \quad \|\nabla Q\|^2 \le \mathrm{e}^{Ct}(\|P_0\|^2 + \|\nabla Q_0\|^2 + Ct), \tag{4.1}$$

for C is a positive constant.

Proof Firstly we differentiate equation (1.2) with respect to x once and set

$$W = \nabla Q, \tag{4.2}$$

then equations (1.1) and (1.2) can be rewritten as

$$\partial_t P = \xi P + (1+i\mu)\Delta P - (1+i\nu)|P|^{2\sigma}P - \nabla P W - r_1 P\nabla W - gr_2 P\nabla\Delta W, \tag{4.3}$$

$$\partial_t W = -\Delta W - g\Delta^2 W + \delta\Delta^3 W - W\nabla W - \eta\nabla(|P|^2). \tag{4.4}$$

Multiplying (4.3) by $\overline{P}$, integrating with respect to x over Ω and taking the real part, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|P\|^2 &= \mathrm{Re}\int_\Omega P_t\overline{P}\mathrm{d}x \\ &= \xi\|P\|^2 - \|\nabla P\|^2 - \int_\Omega |P|^{2\sigma+2}\mathrm{d}x - \mathrm{Re}\int_\Omega \nabla P\overline{P}W\mathrm{d}x \\ &\quad -r_{1r}\int_\Omega |P|^2\nabla W\mathrm{d}x - gr_{2r}\int_\Omega |P|^2\nabla\Delta W\mathrm{d}x,\end{aligned} \tag{4.5}$$

where

$$-\mathrm{Re}\int_\Omega \nabla P\overline{P}W\mathrm{d}x = \frac{1}{2}\int_\Omega |P|^2\nabla W\mathrm{d}x. \tag{4.6}$$

On the other hand, multiplying (4.4) by W and integrating over Ω, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|W\|^2 &= \int_\Omega W_t W\mathrm{d}x \\ &= \|\nabla W\|^2 - g\|\Delta W\|^2 - \delta\|\nabla\Delta W\|^2 - \int_\Omega W^2\nabla W\mathrm{d}x - \eta\int_\Omega \nabla(|P|^2)W\mathrm{d}x,\end{aligned} \tag{4.7}$$

where

$$\int_\Omega W^2\nabla W\mathrm{d}x = \frac{1}{3}\int_\Omega \nabla W^3\mathrm{d}x = 0, \tag{4.8}$$

and

$$-\eta\int_\Omega \nabla(|P|^2)W\mathrm{d}x = \eta\int_\Omega |P|^2\nabla(W)\mathrm{d}x. \tag{4.9}$$

Adding (4.5) and (4.7) together, and noticing (4.6), (4.8) and (4.9), we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|P\|^2+\|W\|^2)=2\xi\|P\|^2-2\|\nabla P\|^2-2\int_\Omega |P|^{2\sigma+2}\mathrm{d}x+2\|\nabla W\|^2$$
$$-2g\|\Delta W\|^2-2\delta\|\nabla\Delta W\|^2+(1+2\eta-2r_{1,r})\int_\Omega |P|^2\nabla W\mathrm{d}x$$
$$-2gr_{2r}\int_\Omega |P|^2\nabla\Delta W\mathrm{d}x. \tag{4.10}$$

According to Young's inequality (2.2) and Hölder's inequality, we have

$$\left|(1+2\eta-2r_{1,r})\int_\Omega |P|^2\nabla W\mathrm{d}x\right|\le |1+2\eta-2r_{1,r}|\left(\int_\Omega |P|^4\mathrm{d}x\right)^{\frac{1}{2}}\|\nabla W\|$$
$$\le\frac{1}{2}\int_\Omega |P|^4\mathrm{d}x+\frac{|1+2\eta-2r_{1,r}|^2}{2}\|\nabla W\|^2$$
$$\le\frac{1}{2}\int_\Omega |P|^{2\sigma+2}\mathrm{d}x+C+\frac{|1+2\eta-2r_{1,r}|^2}{2}\|\nabla W\|^2, \tag{4.11}$$

and

$$\left|-2gr_{2r}\int_\Omega |P|^2\nabla\Delta W\mathrm{d}x\right|\le |2gr_{2r}|\left(\int_\Omega |P|^4\mathrm{d}x\right)^{\frac{1}{2}}\|\nabla\Delta W\|$$
$$\le\frac{\delta}{2}\|\nabla\Delta W\|^2+\frac{|2gr_{2r}|^2}{2\delta}\int_\Omega |P|^4\mathrm{d}x$$
$$\le\frac{\delta}{2}\|\nabla\Delta W\|^2+\frac{1}{2}\int_\Omega |P|^{2\sigma+2}\mathrm{d}x+C. \tag{4.12}$$

Combining (4.10)-(4.12) and $|\xi|=1$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|P\|^2+\|W\|^2)\le 2\|P\|^2+\Big(1+\frac{|1+2\eta-2r_{1,r}|^2}{2}\Big)\|\nabla W\|^2-2\|\nabla P\|^2$$
$$-\int_\Omega |P|^{2\sigma+2}\mathrm{d}x-2g\|\Delta W\|^2-\frac{3}{2}\delta\|\nabla\Delta W\|^2+C. \tag{4.13}$$

Using Gagliardo-Nirenberg inequality (2.3), we deduce

$$\Big(1+\frac{|1+2\eta-2r_{1,r}|^2}{2}\Big)\|\nabla W\|^2\le\Big(1+\frac{|1+2\eta-2r_{1,r}|^2}{2}\Big)\|\nabla\Delta W\|^{\frac{2}{3}}\|W\|^{\frac{4}{3}}$$
$$\le\frac{\delta}{2}\|\nabla\Delta W\|^2+C\|W\|^2. \tag{4.14}$$

By (4.13) and (4.14), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|P\|^2+\|W\|^2)+\int_\Omega |P|^{2\sigma+2}\mathrm{d}x+2g\|\Delta W\|^2+\delta\|\nabla\Delta W\|^2\le C(\|P\|^2+\|W\|^2)+C, \tag{4.15}$$

then we deduce from Gronwall's inequality that

$$\|P\|^2+\|W\|^2\leq \mathrm{e}^{Ct}(\|P_0\|^2+\|W_0\|^2+Ct). \tag{4.16}$$

Lemma 4.2 *Under the assumptions of Lemma* 3.1, *we have*

$$\begin{aligned}&\frac{1}{2\sigma+2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |P|^{2\sigma+2}\mathrm{d}x+\frac{1}{8}\int_\Omega |P|^{4\sigma+2}\\ \leq& -\frac{1}{4}\int_\Omega |P|^{2\sigma-2}[(1+2\sigma)\big|\nabla|P|^2\big|^2-2\mu\sigma\nabla|P|^2\cdot i(P\nabla\overline{P}-\overline{P}\nabla P)+|P\nabla\overline{P}-\overline{P}\nabla P|^2]\\ &+\frac{1}{8}\|\Delta P\|^2+\frac{1}{3}\delta\|\nabla\Delta^2 W\|^2+C,\end{aligned} \tag{4.17}$$

where C is a positive constant.

Proof Since

$$\begin{aligned}\frac{1}{2\sigma+2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |P|^{2\sigma+2}\mathrm{d}x=&\ \xi\|P\|^{2\sigma+2}+\mathrm{Re}\int_\Omega(1+i\mu)|P|^{2\sigma}\overline{P}\Delta P\mathrm{d}x-\int_\Omega|P|^{4\sigma+2}\mathrm{d}x\\ &-\mathrm{Re}\int_\Omega |P|^{2\sigma}\overline{P}\nabla PW\mathrm{d}x+r_{1,r}\int_\Omega|P|^{2\sigma+2}W\mathrm{d}x\\ &+gr_{2,r}\int_\Omega |P|^{2\sigma+2}\nabla\Delta W\mathrm{d}x,\end{aligned} \tag{4.18}$$

for the second term of RSH of (4.18), we have

$$\begin{aligned}\mathrm{Re}\int_\Omega(1+i\mu)|P|^{2\sigma}\overline{P}\Delta P\mathrm{d}x=&-\frac{1}{4}\int_\Omega |P|^{2\sigma-2}[(1+2\sigma)\big|\nabla|P|^2\big|^2\\ &-2\mu\sigma\nabla|P|^2\cdot i(P\nabla\overline{P}-\overline{P}\nabla P)+|P\nabla\overline{P}-\overline{P}\nabla P|^2],\end{aligned} \tag{4.19}$$

for the remaining four terms, using Young's and Gagliargo-Nirenberg inequalities, we have

$$\begin{aligned}\left|\mathrm{Re}\int_\Omega |P|^{2\sigma}\overline{P}\nabla PW\mathrm{d}x\right|\leq&\ \frac{1}{4}\int_\Omega|P|^{4\sigma+2}+C\int_\Omega|\nabla P|^2|W|^2\mathrm{d}x\\ \leq&\ \frac{1}{4}\int_\Omega|P|^{4\sigma+2}+C\|\nabla P\|^2\|W\|_\infty^2\\ \leq&\ \frac{1}{4}\int_\Omega|P|^{4\sigma+2}+C\|\Delta P\|\|P\|\|\nabla\Delta^2W\|^{\frac{n}{5}}\|W\|^{\frac{10-n}{5}}\\ \leq&\ \frac{1}{4}\int_\Omega|P|^{4\sigma+2}+\frac{1}{8}\|\Delta P\|^2+C\|\nabla\Delta^2W\|^{\frac{2n}{5}}\\ \leq&\ \frac{1}{4}\int_\Omega|P|^{4\sigma+2}+\frac{1}{8}\|\Delta P\|^2+\frac{1}{9}\delta\|\nabla\Delta^2W\|^2+C,\end{aligned} \tag{4.20}$$

$$\begin{aligned}\left|r_{1,r}\int_\Omega |P|^{2\sigma+2}W\mathrm{d}x\right| &\le \frac{1}{4}\int_\Omega |P|^{4\sigma+2}+C\int_\Omega |P|^2|W|^2\mathrm{d}x\\ &\le \frac{1}{4}\int_\Omega |P|^{4\sigma+2}+C\|P\|^2\|W\|_\infty^2\\ &\le \frac{1}{4}\int_\Omega |P|^{4\sigma+2}+C\|\nabla\Delta^2 W\|^{\frac{n}{5}}\|W\|^{\frac{10-n}{5}}\\ &\le \frac{1}{4}\int_\Omega |P|^{4\sigma+2}+\frac{1}{9}\delta\|\nabla\Delta^2 W\|^2+C, \end{aligned}\tag{4.21}$$

$$\begin{aligned}\left|gr_{2,r}\int_\Omega |P|^{2\sigma+2}\nabla\Delta W\mathrm{d}x\right| &\le \frac{1}{4}\int_\Omega |P|^{4\sigma+2}+C\int_\Omega |P|^2|\nabla\Delta W|^2\mathrm{d}x\\ &\le \frac{1}{4}\int_\Omega |P|^{4\sigma+2}+C\|P\|^2\|\nabla\Delta W\|_\infty^2\\ &\le \frac{1}{4}\int_\Omega |P|^{4\sigma+2}+C\|\nabla\Delta W\|_\infty^2\\ &\le \frac{1}{4}\int_\Omega |P|^{4\sigma+2}+C\|\nabla\Delta^2 W\|^{\frac{n+6}{5}}\|W\|^{\frac{4-n}{5}}\\ &\overset{(n\le 4)}{\le} \frac{1}{4}\int_\Omega |P|^{4\sigma+2}+\frac{1}{9}\delta\|\nabla\Delta^2 W\|^2+C, \end{aligned}\tag{4.22}$$

and

$$|\xi\|P\|^{2\sigma+2}|\le \frac{1}{8}\int_\Omega |P|^{4\sigma+2}+C.\tag{4.23}$$

Combing the above estimates, we complete the proof.

Lemma 4.3 *Let $P_0(x)\in H^1_{per}(\Omega)\cap L^{2\sigma+2}_{per}(\Omega)$, $Q_0(x)\in H^3_{per}(\Omega)$ and suppose that $\sigma>1$ and Ω is a bounded domain with $\partial\Omega$ in C^m. Then*

$$\|\nabla P\|_2\le \mathrm{e}^{Ct}(\|\nabla P_0\|_2+\|\Delta Q_0\|_2+\|\nabla\Delta Q_0\|_2+\|P_0\|_{2\sigma+2}^{2\sigma+2}+Ct),\tag{4.24}$$

$$\|\Delta Q\|_2+\|\nabla\Delta Q\|_2\le \mathrm{e}^{Ct}(\|\nabla P_0\|_2+\|\Delta Q_0\|_2+\|\nabla\Delta Q_0\|_2+\|P_0\|_{2\sigma+2}^{2\sigma+2}+Ct),\tag{4.25}$$

$$\|P\|_{2\sigma+2}^{2\sigma+2}\le \mathrm{e}^{Ct}(\|\nabla P_0\|_2+\|\Delta Q_0\|_2+\|\nabla\Delta Q_0\|_2+\|P_0\|_{2\sigma+2}^{2\sigma+2}+Ct),\tag{4.26}$$

where C is a positive constant.

Proof Multiplying (4.3) by $-\Delta\overline{P}$, integrating with respect to x over Ω and taking the real part, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla P\|^2 &= \xi\|\nabla P\|^2-\|\Delta P\|^2-\mathrm{Re}\int_\Omega (1+i\nu)|P|^{2\sigma}P\Delta\overline{P}\mathrm{d}x+\mathrm{Re}\int_\Omega \nabla P\Delta\overline{P}W\mathrm{d}x\\ &\quad+\mathrm{Re}\int_\Omega r_1P\Delta\overline{P}\nabla W\mathrm{d}x+\mathrm{Re}\int_\Omega gr_2P\Delta\overline{P}\nabla\Delta W\mathrm{d}x. \end{aligned}\tag{4.27}$$

Multiplying (4.4) by $-\Delta W$ and $-\Delta^2 W$, and integrating over Ω respectively, we deduce

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla W\|^2=\|\Delta W\|^2-g\|\nabla\Delta W\|^2-\delta\|\Delta^2 W\|^2 +\int_\Omega W\nabla W\Delta W\mathrm{d}x+\eta\int_\Omega\nabla(|P|^2)\Delta W\mathrm{d}x, \tag{4.28}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta W\|^2=\|\nabla\Delta W\|^2-g\|\Delta^2 W\|^2-\delta\|\nabla\Delta^2 W\|^2 -\int_\Omega W\nabla W\Delta^2 W\mathrm{d}x-\eta\int_\Omega\nabla(|P|^2)\Delta^2 W\mathrm{d}x. \tag{4.29}$$

Adding (4.27), (4.28) and (4.29), we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\|\nabla P\|^2+\frac{1}{2}\|\nabla W\|^2+\frac{1}{2}\|\Delta W\|^2\Big)\\ &=\xi\|\nabla P\|^2-\|\Delta P\|^2+\|\Delta W\|^2-(g-1)\|\nabla\Delta W\|^2-(\delta+g)\|\Delta^2 W\|^2-\delta\|\nabla\Delta^2 W\|^2\\ &\quad+\mathrm{Re}\int_\Omega(1+i\nu)|P|^{2\sigma}P\Delta\overline{P}\mathrm{d}x+\mathrm{Re}\int_\Omega\nabla P\Delta\overline{P}W\mathrm{d}x+\mathrm{Re}\int_\Omega r_1P\Delta\overline{P}\nabla W\mathrm{d}x\\ &\quad+\mathrm{Re}\int_\Omega gr_2P\Delta\overline{P}\nabla\Delta W\mathrm{d}x+\int_\Omega W\nabla W\Delta W\mathrm{d}x+\eta\int_\Omega\nabla(|P|^2)\Delta W\mathrm{d}x\\ &\quad-\int_\Omega W\nabla W\Delta^2 W\mathrm{d}x-\eta\int_\Omega\nabla(|P|^2)\Delta^2 W\mathrm{d}x.\end{aligned} \tag{4.30}$$

Now we need to control the right hand side of (4.30). Firstly, for the seventh term, we obtain

$$\mathrm{Re}\int_\Omega(1+i\nu)|P|^{2\sigma}P\Delta\overline{P}\mathrm{d}x=-\frac{1}{4}\int_\Omega|P|^{2\sigma-2}[(1+2\sigma)\big|\nabla|P|^2\big|^2 -2\nu\sigma\nabla|P|^2\cdot i(\overline{P}\nabla P-P\nabla\overline{P})+|\overline{P}\nabla P-P\nabla\overline{P}|^2]. \tag{4.31}$$

Meanwhile, according to Young' inequality and Gagliardo-Nirenberg inequality, we obtain the following estimates

$$\begin{aligned}&2\|\Delta W\|^2-2(g-1)\|\nabla\Delta W\|^2+2\mathrm{Re}\int_\Omega\nabla P\Delta\overline{P}W\mathrm{d}x+2\mathrm{Re}\int_\Omega r_1P\Delta\overline{P}\nabla W\mathrm{d}x\\ &\quad+\mathrm{Re}\int_\Omega gr_2P\Delta\overline{P}\nabla\Delta W\mathrm{d}x+2\int_\Omega W\nabla W\Delta W\mathrm{d}x\\ &\le 2\|\Delta W\|^2+2|g-1|\|\nabla^4 W\|^{\frac{3}{2}}\|W\|^{\frac{1}{2}}+2\|W\|_\infty\|\nabla P\|\|\Delta P\|+2|r_1|\|\nabla W\|_\infty\|P\|\|\Delta P\|\\ &\quad+2|r_2|\|\nabla\Delta W\|_\infty\|P\|\|\Delta P\|+2\|\nabla W\|_\infty\|W\|\|\Delta W\|\\ &\le 2\|\Delta W\|^2+\frac{\delta+g}{4}\|\nabla^4 W\|^2+C+2C\|\nabla^4 W\|^{\frac{n}{8}}\|W\|^{\frac{8-n}{8}}\|\nabla^2 P\|^{\frac{1}{2}}\|P\|^{\frac{1}{2}}\|\Delta P\|\\ &\quad+2|r_1|\|P\|\|\Delta P\|\|\nabla^5 W\|^{\frac{1+n/2}{5}}\|W\|^{\frac{4-n/2}{5}}+2|r_2|\|P\|\|\Delta P\|\|\nabla^5 W\|^{\frac{3+n/2}{5}}\|W\|^{\frac{2-n/2}{5}}\\ &\quad+2\|W\|\|\Delta W\|\|\nabla^5 W\|^{\frac{1+n/2}{5}}\|W\|^{\frac{4-n/2}{5}}\\ &\le 2\|\Delta W\|^2+\frac{\delta+g}{4}\|\Delta^2 W\|^2+C\|\Delta P\|^{\frac{3}{2}\times\frac{16}{16-n}}+C+\frac{1}{8}\|\Delta P\|^2+C\|\nabla^5 W\|^{\frac{2+n}{5}}\end{aligned}$$

$$+\frac{1}{8}\|\Delta P\|^2+C\|\nabla^5 W\|^{\frac{6+n}{5}}+\frac{1}{4}\|\Delta W\|^2+C\|\nabla^5 W\|^{\frac{2+n}{5}}$$
$$\overset{(n\le 4)}{\le}\frac{3}{8}\|\Delta P\|^2+\frac{\delta+g}{4}\|\Delta^2 W\|^2+\frac{1}{3}\delta\|\nabla\Delta^2 W\|^2+\frac{9}{4}\|\Delta W\|^2+C. \qquad (4.32)$$

Next, for the remaining two terms, we obtain

$$\left|2\eta\int_\Omega \nabla(|P|^2)\Delta W\mathrm{d}x\right|=\left|2\eta\int_\Omega |P|^2\nabla\Delta W\mathrm{d}x\right|\le 2\eta\|\nabla\Delta W\|\|P\|_4^2$$
$$\le C\|\nabla^5 W\|^{\frac{3}{5}}\|W\|^{\frac{2}{5}}\|\nabla^2 P\|^{\frac{n}{4}}\|P\|^{\frac{8-n}{4}}$$
$$\le \frac{1}{9}\delta\|\nabla\Delta^2 W\|^2+C\|\nabla^2 P\|^{\frac{n}{4}\times\frac{10}{7}}$$
$$\overset{(n\le 28/5)}{\le}\frac{1}{9}\delta\|\nabla\Delta^2 W\|^2+\frac{1}{8}\|\Delta P\|^2+C, \qquad (4.33)$$

and

$$\left|-2\int_\Omega W\nabla W\Delta^2 W\mathrm{d}x-2\eta\int_\Omega \nabla(|P|^2)\Delta^2 W\mathrm{d}x\right|$$
$$\le 2\left|\int_\Omega W\nabla W\Delta^2 W\mathrm{d}x\right|+2\eta\left|\int_\Omega \nabla(|P|^2)\Delta^2 W\mathrm{d}x\right|$$
$$\le \frac{1}{9}\delta\|\nabla\Delta^2 W\|^2+C(\|W\|_4^4+\|P\|_4^4)$$
$$\le \frac{1}{9}\delta\|\nabla\Delta^2 W\|^2+C(\|\nabla^4 W\|^{\frac{n}{4}}\|W\|^{\frac{16-n}{4}}+\|\nabla^2 P\|^{\frac{n}{2}}\|P\|^{\frac{8-n}{2}})$$
$$\overset{(n\le 4)}{\le}\frac{1}{9}\delta\|\nabla\Delta^2 W\|^2+\frac{\delta+g}{4}\|\Delta^2 W\|^2+\frac{1}{8}\|\Delta P\|^2+C. \qquad (4.34)$$

Combing the above estimates, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla P\|^2+\|\nabla W\|^2+\|\Delta W\|^2)+\frac{1}{4}\|\Delta P\|^2+\frac{\delta+g}{2}\|\Delta^2 W\|^2+\frac{1}{9}\delta\|\nabla\Delta^2 W\|^2$$
$$\le -\frac{1}{4}\int_\Omega |P|^{2\sigma-2}[(1+2\sigma)\big|\nabla|P|^2\big|^2-2\nu\sigma\nabla|P|^2\cdot i(\overline{P}\nabla P-P\nabla\overline{P})$$
$$+|\overline{P}\nabla P-P\nabla\overline{P}|^2]+C(\|\nabla P\|^2+\|\nabla W\|^2+\|\Delta W\|^2)+C, \qquad (4.35)$$

Adding (4.17)and (4.35), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\nabla P\|^2+\frac{1}{2}\|\nabla W\|^2+\frac{1}{2}\|\Delta W\|^2+\frac{1}{2\sigma+2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |P|^{2\sigma+2}\mathrm{d}x\right)$$
$$+\frac{1}{4}\|\Delta P\|^2+\frac{\delta+g}{2}\|\Delta^2 W\|^2+\frac{1}{9}\delta\|\nabla\Delta^2 W\|^2+\frac{1}{8}\int_\Omega |P|^{4\sigma+2}$$
$$\le -\frac{1}{4}\int_\Omega |P|^{2\sigma-2}[2(1+2\sigma)\big|\nabla|P|^2\big|^2-2(\nu-\mu)\sigma\nabla|P|^2\cdot i(\overline{P}\nabla P-P\nabla\overline{P})$$
$$+2|\overline{P}\nabla P-P\nabla\overline{P}|^2]\mathrm{d}x+C(\|\nabla P\|^2+\|\nabla W\|^2+\|\Delta W\|^2)+C. \qquad (4.36)$$

Note that if assumption (A1) holds, then the term $|P|^{2\sigma-2}[2(1+2\sigma)\big|\nabla|P|^2\big|^2-2(\nu-\mu)\sigma\nabla|P|^2\cdot i(\overline{P}\nabla P-P\nabla\overline{P})+2|\overline{P}\nabla P-P\nabla\overline{P}|^2]$ is positive, therefore we can omit this term and then integrating with respect to t to get the final estimate

$$\begin{aligned}&\|\nabla P\|^2+\|\nabla W\|^2+\|\Delta W\|^2+\|P\|_{2\sigma+2}^{2\sigma+2}\\ &\le \mathrm{e}^{Ct}(\|\nabla P_0\|^2+\|\nabla W_0\|^2+\|\Delta W_0\|^2+\|P_0\|_{2\sigma+2}^{2\sigma+2}+Ct).\end{aligned}\tag{4.37}$$

The following lemma was proved in [18], we omit the proof here.

Lemma 4.4 *Under the assumptions of Lemma* 3.3*, we have*

$$\|P\|_{H^1_{per}}\le C,\quad \|\nabla Q\|_\infty\le C,\tag{4.38}$$

where C is a positive constant.

Lemma 4.5 *Let $s\le\frac{2\sqrt{1+\mu^2}}{\sqrt{1+\mu^2}-1}$, then*

$$\|P\|_s^s\le C,\tag{4.39}$$

where C is a positive constant.

Proof By replacing the exponent $\sigma+2$ in Lemma 4.2 by s, we have

$$\begin{aligned}&\frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|P|^s\mathrm{d}x+\frac{1}{8}\int_\Omega|P|^{s+2\sigma}\\ &\le-\frac{1}{4}\int_\Omega|P|^{s-4}[(s-1)\big|\nabla|P|^2\big|^2-2\mu(s-2)\nabla|P|^2\cdot i(P\nabla\overline{P}-\overline{P}\nabla P)\\ &\quad+|P\nabla\overline{P}-\overline{P}\nabla P|^2]\mathrm{d}x+\frac{1}{8}\|\Delta P\|^2+\frac{1}{3}\delta\|\nabla\Delta^2W\|^2+C.\end{aligned}\tag{4.40}$$

Note that if the assumption $s\le\frac{2\sqrt{1+\mu^2}}{\sqrt{1+\mu^2}-1}$ holds, then the term $|P|^{s-4}[(s-1)\big|\nabla|P|^2\big|^2-2\mu(s-2)\nabla|P|^2\cdot i(P\nabla\overline{P}-\overline{P}\nabla P)+|P\nabla\overline{P}-\overline{P}\nabla P|^2]$ is positive, therefore we can omit this term. By (4.40), (4.36) and Gronwall's inequality, we complete the proof.

Lemma 4.6 *Let $P_0(x)\in H^2_{per}(\Omega)$, $Q_0(x)\in H^4_{per}(\Omega)$ and suppose that $\sigma>1$ and Ω is a bounded domain with $\partial\Omega$ in C^m. Then*

$$\|\Delta P\|^2+\|\Delta^2 Q\|^2\le\mathrm{e}^{Ct}(\|\Delta P_0\|^2+\|\Delta^2 Q_2\|^2+ct),\tag{4.41}$$

where C is a positive constant.

Proof Multiplying (4.3) by $\Delta^2\overline{P}$, integrating over Ω and taking the real part, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta P\|^2=&\,\xi\|\Delta P\|^2-\|\nabla\Delta P\|^2-\mathrm{Re}\int_\Omega(1+i\nu)|P|^{2\sigma}P\Delta^2\overline{P}\mathrm{d}x-\mathrm{Re}\int_\Omega\nabla P\Delta^2\overline{P}W\mathrm{d}x\\ &-\mathrm{Re}\int_\Omega r_1P\Delta^2\overline{P}\nabla W\mathrm{d}x-\mathrm{Re}\int_\Omega gr_2P\Delta^2\overline{P}\nabla\Delta W\mathrm{d}x.\end{aligned}\tag{4.42}$$

Multiplying (4.4) by $-\Delta^3 W$ and integrating over Ω, we deduce

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta W\|^2 =& \|\Delta^2 W\|^2 - g\|\nabla\Delta^2 W\|^2 - \delta\|\Delta^3 W\|^2\\ &+\int_\Omega W\nabla W\Delta^3 W\mathrm{d}x + \eta\int_\Omega \nabla(|P|^2)\Delta^3 W\mathrm{d}x. \end{aligned}\tag{4.43}$$

Adding the above two equalities yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|\Delta P\|^2 + \|\nabla\Delta W\|^2)\\ =& 2\xi\|\Delta P\|^2 - 2\|\nabla\Delta P\|^2 + 2\|\Delta^2 W\|^2 - 2g\|\nabla\Delta^2 W\|^2 - 2\delta\|\Delta^3 W\|^2\\ &-2\mathrm{Re}\int_\Omega (1+i\nu)|P|^{2\sigma}P\Delta^2\overline{P}\mathrm{d}x - 2\mathrm{Re}\int_\Omega \nabla P\Delta^2\overline{P}W\mathrm{d}x\\ &-2\mathrm{Re}\int_\Omega r_1 P\Delta^2\overline{P}\nabla W\mathrm{d}x - 2\mathrm{Re}\int_\Omega gr_2 P\Delta^2\overline{P}\nabla\Delta W\mathrm{d}x\\ &+2\int_\Omega W\nabla W\Delta^3 W\mathrm{d}x + 2\eta\int_\Omega \nabla(|P|^2)\Delta^3 W\mathrm{d}x. \end{aligned}\tag{4.44}$$

According to Gagliardo-Nirenberg inequality, Lemmas 4.1, 4.3 and 4.5 and assumption (A2), we have

$$\begin{aligned}\left|-2\mathrm{Re}\int_\Omega (1+i\nu)|P|^{2\sigma}P\Delta^2\overline{P}\mathrm{d}x\right| \le& 6|1+i\nu|\int_\Omega |P|^{2\sigma}|\nabla P||\nabla\Delta P|\mathrm{d}x\\ \le& \frac{1}{3}\|\nabla\Delta P\|^2 + C\||P|^{2\sigma}\|_4^2\|\nabla P\|_4^2\\ \le& \frac{1}{3}\|\nabla\Delta P\|^2 + C\|\nabla\Delta P\|^{\frac{n}{4}}\|\nabla P\|^{\frac{8-n}{4}}\\ \le& \frac{2}{3}\|\nabla\Delta P\|^2 + C, \end{aligned}\tag{4.45}$$

and

$$\begin{aligned}\left|-2\mathrm{Re}\int_\Omega \nabla P\Delta^2\overline{P}W\mathrm{d}x\right| =& \left|2\mathrm{Re}\int_\Omega \nabla P\nabla\Delta\overline{P}\nabla W\mathrm{d}x + 2\mathrm{Re}\int_\Omega \Delta P\nabla\Delta\overline{P}W\mathrm{d}x\right|\\ \le& 2\|\nabla W\|_\infty\|\nabla P\|\|\nabla\Delta P\| + 2\|W\|_\infty\|\Delta P\|\|\nabla\Delta P\|\\ \le& \frac{1}{3}\|\nabla\Delta P\|^2 + C\|\nabla\Delta W\|^{\frac{n}{2}}\|\nabla W\|^{\frac{4-n}{2}} + C\|\Delta P\|^2\\ \overset{(n\le 4)}{\le}& \frac{1}{3}\|\nabla\Delta P\|^2 + C\|\nabla\Delta W\|^2 + C\|\Delta P\|^2 + C. \end{aligned}\tag{4.46}$$

For the remaining terms of the RHS of (4.44), we have

$$
\begin{aligned}
&\left|2\|\Delta^2W\|^2-2\mathrm{Re}\int_\Omega r_1P\Delta^2\overline{P}\nabla W\mathrm{d}x-2\mathrm{Re}\int_\Omega gr_2P\Delta^2\overline{P}\nabla\Delta W\mathrm{d}x\right|\\
\le{}& 2\|\Delta^2W\|^2+\left|2\mathrm{Re}\int_\Omega r_1(P\nabla\Delta\overline{P}\Delta W+\nabla P\nabla\Delta\overline{P}\nabla W)\mathrm{d}x\right|\\
&+\left|2\mathrm{Re}\int_\Omega gr_2(P\nabla\Delta\overline{P}\Delta^2W+\nabla P\nabla\Delta\overline{P}\nabla\Delta W)\mathrm{d}x\right|\\
\le{}& 2\|\Delta^2W\|^2+2|r_1|\|P\|\|\nabla\Delta P\|\|\Delta W\|_\infty+2|r_1|\|\nabla P\|\|\nabla\Delta P\|\|\nabla W\|_\infty\\
&+2g|r_2|\|P\|\|\nabla\Delta P\|\|\Delta_2W\|_\infty+2g|r_2|\|\nabla P\|\|\nabla\Delta P\|\|\nabla\Delta W\|_\infty\\
\le{}& 2\|\Delta^2W\|^2+\frac{1}{3}\|\nabla\Delta P\|^2+C\|\nabla W\|_{H^3}^{\frac{2+n}{3}}\|\nabla W\|^{\frac{4-n}{3}}+C\|\nabla W\|_{H^2}^{\frac{n}{2}}\|\nabla W\|^{\frac{4n}{2}}\\
&+C\|\nabla W\|_{H^5}^{\frac{6+n}{5}}\|\nabla W\|^{\frac{4n}{5}}+C\|\nabla W\|_{H^4}^{\frac{4+n}{4}}\|\nabla W\|^{\frac{4n}{4}}\\
\le{}& \frac{1}{3}\|\nabla\Delta P\|^2+g\|\nabla\Delta^2W\|^2+\frac{\delta}{4}\|\Delta^3W\|^2+C\|\nabla\Delta W\|^2+C,
\end{aligned}
\tag{4.47}
$$

and

$$
\begin{aligned}
&\left|2\int_\Omega W\nabla W\Delta^3W\mathrm{d}x+2\eta\int_\Omega\nabla(|P|^2)\Delta^3W\mathrm{d}x\right|\\
\le{}& 2\|W\|_\infty\|\nabla W\|\|\Delta^3W\|+4\eta\|P\|_\infty\|\nabla P\|\|\Delta^3W\|\\
\le{}& \frac{\delta}{2}\|\Delta^3W\|+C.
\end{aligned}
\tag{4.48}
$$

Then combining (4.44)-(4.48) and noticing $|\xi|=1$, we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}(\|\Delta P\|^2+\|\nabla\Delta W\|^2)\le C(\|\Delta P\|^2+\|\nabla\Delta W\|^2)+C. \tag{4.49}
$$

Lemma 4.7 *Let $P_0(x)\in H^2_{per}(\Omega)$, $Q_0(x)\in H^4_{per}(\Omega)$ and suppose that $\sigma>1$ and Ω is a bounded domain with $\partial\Omega$ in C^m. Then*

$$
\|P\|_\infty\le C,\quad \|\Delta Q\|_\infty\le C, \tag{4.50}
$$

where C is a positive constant.

Lemma 4.8 *Let $P_0(x)\in H^2_{per}(\Omega)$, $Q_0(x)\in H^4_{per}(\Omega)$ and suppose that $\sigma>1$ be a bounded domain with $\partial\Omega$ in C^m. Then*

$$
\|P\|_{H^2_{per}}\le C,\quad \|Q\|_{H^4_{per}}\le C, \tag{4.51}
$$

for C is a positive constant.

For the proof of Lemmas 4.7 and 4.8 in detail, one can refer to [18].

Lemma 4.9 *Let $P_0(x)\in H^3_{per}(\Omega)$, $Q_0(x)\in H^5_{per}(\Omega)$ and suppose that $\sigma>1$ and Ω is a bounded domain with $\partial\Omega$ in C^m. Then*

$$\|\nabla\Delta P\|+\|\nabla\Delta^2 Q\|\le \mathrm{e}^{Ct}(\|\nabla\Delta P_0\|+\|\nabla\Delta^2 Q_0\|+Ct),\tag{4.52}$$

where C is a positive constant.

Proof Multiplying (4.3) by $-\Delta^3\overline{P}$, integrating over Ω and taking the real part, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta P\|^2=&\ \xi\|\nabla\Delta P\|^2-\|\Delta^2 P\|^2-\mathrm{Re}\int_\Omega(1+i\nu)|P|^{2\sigma}P\Delta^3\overline{P}\mathrm{d}x\\&-\mathrm{Re}\int_\Omega\nabla P\Delta^3\overline{P}W\mathrm{d}x-\mathrm{Re}\int_\Omega r_1P\Delta^3\overline{P}\nabla W\mathrm{d}x\\&-\mathrm{Re}\int_\Omega gr_2P\Delta^3\overline{P}\nabla\Delta W\mathrm{d}x,\end{aligned}\tag{4.53}$$

Multiplying (4.4) by $\Delta^4 W$ and integrating over Ω, we deduce

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta^2 W\|^2=&\ \|\nabla\Delta^2 W\|^2-g\|\Delta^3 W\|^2-\delta\|\Delta^3 W\|^2\\&-\int_\Omega W\nabla W\Delta^4 W\mathrm{d}x-\eta\int_\Omega\nabla(|P|^2)\Delta^4 W\mathrm{d}x.\end{aligned}\tag{4.54}$$

Adding the above two equalities arrives at

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla\Delta P\|^2+\|\Delta^2 W\|^2)\\=&\ 2\xi\|\nabla\Delta P\|^2-2\|\Delta^2 P\|^2+2\|\nabla\Delta^2 W\|^2-2g\|\Delta^3 W\|^2-2\delta\|\Delta^3 W\|^2\\&-2\mathrm{Re}\int_\Omega(1+i\nu)|P|^{2\sigma}P\Delta^3\overline{P}\mathrm{d}x-2\mathrm{Re}\int_\Omega\nabla P\Delta^3\overline{P}W\mathrm{d}x\\&-2\mathrm{Re}\int_\Omega r_1P\Delta^3\overline{P}\nabla W\mathrm{d}x-2\mathrm{Re}\int_\Omega gr_2P\Delta^3\overline{P}\nabla\Delta W\mathrm{d}x\\&-2\int_\Omega W\nabla W\Delta^4 W\mathrm{d}x-2\eta\int_\Omega\nabla(|P|^2)\Delta^4 W\mathrm{d}x,\end{aligned}\tag{4.55}$$

In order to control the RHS of (4.44), using the previous lemmas, we have

$$\begin{aligned}&\left|2\mathrm{Re}\int_\Omega(1+i\nu)|P|^{2\sigma}P\Delta^3\overline{P}\mathrm{d}x\right|\\\le&\ 2|1+i\nu|\int_\Omega(6|\nabla P|^2|P|+3|P|^2|\Delta P|)|P|^{2\sigma-2}|\Delta^2 P|\mathrm{d}x\\\le&\ \frac{1}{8}\|\Delta^2 P\|^2+C(\||\nabla P|^4\|\|P\|_\infty^2+\||\Delta P|^2\|\|P\|_\infty^4)\||P|^{4\sigma-4}\|\\\le&\ \frac{1}{8}\|\Delta^2 P\|^2+C(\|\nabla P\|_8^4+\|\Delta P\|_4^2)\\\le&\ \frac{1}{8}\|\Delta^2 P\|^2+C(\|\Delta^2\nabla P\|^{\frac{n}{2}}\|\nabla P\|^{\frac{8-n}{2}}+\|\Delta^2 P\|^{\frac{n}{4}}\|\Delta P\|^{\frac{8-n}{4}})\\\le&\ \frac{3}{8}\|\Delta^2 P\|^2+C,\end{aligned}\tag{4.56}$$

$$\begin{aligned}
&\left|2\mathrm{Re}\int_\Omega \nabla P\Delta^3\overline{P}W\mathrm{d}x\right|\\
=&\left|2\mathrm{Re}\int_\Omega (\Delta W\nabla P+2\Delta P\nabla W+W\nabla\Delta P)\Delta^2\overline{P}\mathrm{d}x\right|\\
\le& 2(\|\Delta W\|\|\nabla P\|_\infty+2\|\Delta P\|\|\nabla W\|_\infty+\|\nabla\Delta P\|\|W\|_\infty)\|\Delta^2 P\|\\
\le&\frac18\|\Delta^2P\|^2+C\|\nabla\Delta P\|^{\frac n2}\|P\|^{\frac{4-n}{2}}+C\|\nabla\Delta P\|^2+C\\
\le&\frac18\|\Delta^2P\|^2+C\|\nabla\Delta P\|^2+C,
\end{aligned}\tag{4.57}$$

and

$$\begin{aligned}
&\left|2\mathrm{Re}\int_\Omega r_1P\Delta^3\overline{P}\nabla W\mathrm{d}x\right|\\
=&\left|2\mathrm{Re}\int_\Omega (\Delta P\nabla W+2\Delta W\nabla P+P\nabla\Delta W)\Delta^2\overline{P}\mathrm{d}x\right|\\
\le& 2(\|\Delta P\|\|\nabla W\|_\infty+2\|\Delta W\|\|\nabla P\|_\infty+\|\nabla\Delta W\|\|P\|_\infty)\|\Delta^2 P\|\\
\le&\frac18\|\Delta^2P\|^2+C\|\nabla\Delta P\|^{\frac n2}\|P\|^{\frac{4-n}{2}}+C\|\nabla\Delta P\|^2+C\\
\le&\frac18\|\Delta^2P\|^2+C\|\nabla\Delta P\|^2+C.
\end{aligned}\tag{4.58}$$

For the remaining terms, using Gagliardo-Nirenberg inequality and the previous estimates, we obtain

$$\begin{aligned}
&\left|2\|\nabla\Delta^2W\|^2+2\mathrm{Re}\int_\Omega gr_2P\Delta^3\overline{P}\nabla\Delta W\mathrm{d}x\right|\\
=&2\|\nabla\Delta^2W\|^2+\left|2\mathrm{Re}\int_\Omega gr_2(\nabla\Delta^2WP+2\Delta^2W\nabla P+\nabla\Delta W\Delta P)\Delta^2\overline{P}\mathrm{d}x\right|\\
\le& 2\|\nabla\Delta^2W\|^2+2g|r_2|(\|\nabla\Delta^2W\|\|P\|_\infty+2\|\nabla P\|\|\Delta^2W\|_\infty+\|\nabla\Delta W\|_\infty\|\Delta P\|)\|\Delta^2P\|\\
\le&\frac18\|\Delta^2P\|^2+2\|\nabla\Delta^2W\|^2+C\|\Delta^3W\|^{\frac53}\|W\|^{\frac13}\\
&+C\|\Delta^3W\|^{\frac{4+n}{4}}\|\Delta W\|^{\frac{4-n}{4}}+C\|\nabla\Delta^2W\|^{\frac{4+n}{4}}\|\nabla W\|^{\frac{4-n}{4}}\\
\le&\frac18\|\Delta^2P\|^2+g\|\Delta^3W\|^2+C\|\Delta^2W\|^2+C.
\end{aligned}\tag{4.59}$$

Combing the above estimates yields that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla\Delta P\|^2+\|\Delta^2W\|^2)\le C(\|\nabla\Delta P\|^2+\|\Delta^2W\|^2)+C.\tag{4.60}$$

Therefore, from the above lemmas, we get the following lemma.

Lemma 4.10 *Let $P_0(x) \in H^3_{per}(\Omega)$, $Q_0(x) \in H^5_{per}(\Omega)$ and suppose that $\sigma > 1$ and Ω is a bounded domain with $\partial\Omega$ in C^m. Then*

$$\|P\|_{H^3} + \|Q\|_{H^5} \le C, \tag{4.61}$$

where C is a positive constant.

5 The Local Solutions and Global Solutions

In this section, we will obtain the existence and uniqueness of the local solutions and global solutions of the periodic initial value problem (1.1)-(1.4). From the lemmas in Section 3, we deduce our main result:

Theorem 5.1(Local existence) *Assume that $P_0(x) \in H^3_{per}(\Omega)$, $Q_0(x) \in H^5_{per}(\Omega)$ and the parameter σ, μ, ν satisfy assumptions* (A1) *and* (A2)*. Then there exist local solutions $P(x,t)$ and $Q(x,t)$ to the periodic initial value problem* (1.1)-(1.4)*, satisfying*

$$P(x,t) \in C\big((0,t_0); H^3_{per}(\Omega)\big), \quad Q(x,t) \in C\big((0,t_0); H^5_{per}(\Omega)\big),$$

where t_0 depends on $\|P_0\|_{H^3_{per}}$ and $\|Q_0\|_{H^5_{per}}$.

Finally, we are able to deduce from this local existence theorem combined with the a priori estimates that the solutions exist globally in time.

Theorem 5.2(Global existence) *Assume that $P_0(x) \in H^3_{per}(\Omega)$, $Q_0(x) \in H^5_{per}(\Omega)$ and the parameter σ, μ, ν satisfy assumptions* (A1) *and* (A2)*. Then there exist global solutions $P(x,t)$ and $Q(x,t)$ to the periodic initial value problem* (1.1)-(1.4)*, satisfying*

$$P(x,t) \in C\big((0,\infty); H^3_{per}(\Omega)\big), \quad Q(x,t) \in C\big((0,\infty); H^5_{per}(\Omega)\big),$$

where the periodic initial value problem (1.1)-(1.4)*.*

Proof From Theorem 5.1, there exist local solutions $P(x,t)$ and $Q(x,t)$ of the periodic initial value problem (1.1)-(1.4) and the existence time t_0 depends on $\|P_0(x)\|_{H^3_{per}}$ and $\|Q_0(x)\|_{H^5_{per}}$. According to the priori estimates in Section 4 and the so-called continuity method, we complete the proof.

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