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SMITH NORMAL FORMAL OF DISTANCE MATRIX OF BLOCK GRAPHS*†

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Abstract

A connected graph, whose blocks are all cliques (of possibly varying sizes), is called a block graph. Let $D(G)$ be its distance matrix. In this note, we prove that the Smith normal form of $D(G)$ is independent of the interconnection way of blocks and give an explicit expression for the Smith normal form in the case that all cliques have the same size, which generalize the results on determinants.

Keywords block graph; distance matrix; Smith normal form

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1 Introduction

Let G be a connected graph (or strong connected digraph) with vertex set $\{1, 2, \cdots, n\}$. The *distance matrix* $D(G)$ is an $n \times n$ matrix in which $d_{i,j} = d(i, j)$ denotes the distance from vertex i to vertex j. Like the adjacency matrix and Laplacian matrix of a graph, $D(G)$ is also an integer matrix and there are many results on distance matrices and their applications.

For distance matrices, Graham and Pollack [10] proved a remarkable result that gives a formula of the determinant of the distance matrix of a tree depending only on the number n of vertices of the tree. The determinant is given by $\det D = (-1)^{n-1}(n-1)2^{n-2}$. This result has attracted much interest in algebraic graph theory. Graham, Hoffman and Hosoya [8] showed that the determinant of the distance matrix $D(G)$ of a strong connected directed graph G is a function of the distance matrix of its strong blocks:

Theorem 1.1 *If G is a strong connected digraph with strong blocks $G_1, G_2, \cdots, G_r$, then*

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$$Cof(D(G)) = \prod_{i=1}^{r} Cof(D(G_i)), \quad and \quad \det(D(G)) = \sum_{i=1}^{r} \det(D(G_i)) \prod_{j\neq i} Cof(D(G_j)),$$

where $Cof(A)$ is the sum of all cofactors of matrix A.

Graham and Lovász [9] computed the inverse of the distance matrix of a tree and studied the characteristic polynomial of the distance matrix of a tree. For more details about the distance matrix spectrum see [16] as well as the references therein. Almost all results obtained for the distance matrix of trees were extended to the case of weighted trees by Bapat et al. [2], and extended to the case that all blocks are cliques in [5,19]. Extensions were done not only concerning the class of graphs but also regarding the distance matrix itself. Indeed, Bapat et al. [4] generalized the concept and its properties of the distance matrix to q-analogue of the distance matrix. Aouchiche and Hansen [1] investigated the spectrum of two distance Laplacian matrices.

For an $n \times n$ integer matrix A, the *Smith normal form* of A, denoted by $Snf(A)$, is an $n \times n$ diagonal integer matrix

$$S = \mathrm{diag}(s_1, s_2, \cdots, s_n),$$

where $s_1, \cdots, s_n$ are nonnegative integers and $s_i | s_{i+1}$ $(1 \leq i \leq n-1)$ satisfies that there exist invertible integer matrices P, Q such that $PAQ = S$. Since $\det Snf(A) = |\det A|$ and $\mathrm{rank}(Snf(A)) = \mathrm{rank}(A)$, the Smith normal form is more refined invariant than (the absolute value of) the determinant and the rank. The Smith normal form of a matrix has some arithmetic and combinatorial significance and was studied in arithmetic geometry [13], in statistical physics [7] and in combinatorics [3]. There are also interpretations of the critical group in discrete dynamics (chip-firing games and abelian sandpile models [3]). For the Laplacian matrix $L(G)$ of a graph G, the above group has been called the critical group (or sandpile group) of a graph G. And there are a few results on the Smith normal form of Laplacian matrix of a graph [12-14,17]. For the results on the smith normal forms of other matrices of graph, see [6,20,21].

According to Theorem 1.1, the determinant of the distance matrix of graph does not change if the blocks of the graph are reassembled in some other way. Since the Smith normal form of a matrix is a refinement of determinant, the following question naturally arises.

Problem 1.1 Is the Smith normal form of the distance matrix of a connected graph independent of the connection way of its blocks?

In the case of a tree T on n vertices, the blocks are precisely the edge (K_2, the complete graph on two vertices) and $\det D(T) = (-1)^{n-1}(n-1)2^{n-2}$. In [11], it is shown that the Smith normal form of $D(T)$ is $\mathrm{diag}(1, 1, 2, \cdots, 2, 2(n-1))$, which is independent of the structure of the tree T.

A graph G is called *block graph* if all blocks of G are cliques. A tree and its line graph are block graphs. The aim of this paper is to extend the above result on trees to block graphs, also to prove the Smith normal form of the distance matrix $D(G)$ of a block graph G is dependent only on the sizes of its cliques and to obtain the explicit expression of the Smith normal form of the distance matrix $D(G)$ of a block graph G if all blocks have the same size.

2 The Smith Normal Forms of $D(G)$ for Block Graphs

In this section, we investigate the Smith normal form of the distance matrix for a block graph. We first recall some concepts and computing methods about the Smith normal form of an integer matrix. An $n \times n$ integer matrix P is called *unimodular* if $|\det P| = 1$. In other words, the unimodular matrices are precisely those integer matrices with integer inverses. Recall that two $n \times n$ integer matrices A and B are *unimodularly equivalent,* and denoted by $A \sim B$ if there exist unimodular matrices P and Q such that $PAQ = B$.

It is well known that every integer matrix A is unimodularly equivalent to its Smith normal form $S = \mathrm{diag}(s_1, \cdots, s_n)$ and the matrix S is unique and may be obtained from A using (integer) elementary row and column operations which are invertible over the ring of integers:

- swapping any two rows or any two columns;
- adding integer multiples of one row/column to another row/column;
- multiplying any row/column by ± 1.

Moreover, the product $\delta_i(A) = s_1 s_2 \cdots s_i$ equals the nonnegative greatest common divisor (gcd) of all determinants of $i \times i$-submatrices of A. $s_1, s_2, \cdots, s_n$ and $\delta_1, \delta_2, \cdots, \delta_n$ are called the *invariant divisors* and *determinantal divisors* of A respectively, see [18] for details. We can also obtain the Smith normal form of a matrix by the computation of determinantal divisors.

Considering the matrix A as a linear map $\mathrm{Z}^n \to \mathrm{Z}^n$, its *cokernel* has the form $\mathrm{Z}^n/A\mathrm{Z}^n$. It is a finitely generated abelian group. By the structure theorem for finitely generated abelian groups, we have

$$\mathrm{Z}^n/A\mathrm{Z}^n \cong \prod_{i=1}^{n} \mathrm{Z}/s_i\mathrm{Z}.$$

(Of course, $\mathrm{Z}/1\mathrm{Z} \cong 0$ is the trivial group and $\mathrm{Z}/0\mathrm{Z} \cong \mathrm{Z}$.) We can also obtain the Smith normal form of a matrix by computing its cokernel.

Let G be a block graph with k cliques $G_1, G_2, \cdots, G_k$, where the clique G_i has n_i vertices, hence the number of vertices G is $n = \sum\limits_{i=1}^{k} n_i - k + 1$. In [5], Bapt and Sivasubramanian showed that

$$\det D(G) = (-1)^{n-1} \sum_{i=1}^{k} \frac{n_i - 1}{n_i} \prod_{i=1}^{k} n_i, \tag{2.1}$$

a formula depending only on the number of vertices of cliques $G_1, G_2, \cdots, G_k$, but not on the interconnection of these blocks. In this section we prove that the Smith normal form of $D(G)$ also depends only on the number of vertices of blocks, but not the interconnection of these blocks.

Let I_m denote an $m \times m$ identity matrix, $J_{m \times n}$ be an $m \times n$ matrix with all entries equaling to 1, and j denote an n-dimensional column vector with all entries equaling to 1, and in general, we omit the size of I, J, j for simplicity. At last let $A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ denote the direct sum of matrices A and B and A^{T} be the transpose of the matrix A.

To do this, we need an elementary result on the Smith normal form of the integer matrix $aI - bJ$.

Let

$$P_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & -1 & 0 \\ 0 & 1 & 0 & \cdots & 0 & -1 & 0 \\ 0 & 0 & 1 & \cdots & 0 & -1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ -1 & -1 & -1 & \cdots & -1 & -1 & -1 \end{pmatrix},$$

$$Q_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ -1 & -1 & -1 & \cdots & -1 & 1-n & -1 \end{pmatrix}. \tag{2.2}$$

Then $P_n^{-1} = Q_n$ and $\det P = \det Q = -1$.

Lemma 2.1 *Let P_n, Q_n be $n \times n$ matrices defined in* (2.2), *and a, b be integers. Then*

$$P_n(aI_n - bJ_n)Q_n = aI_{n-2} \oplus \begin{pmatrix} a & b \\ 0 & a - nb \end{pmatrix}.$$

Theorem 2.1 *Let G be a block graph with cliques $G_1, G_2, \cdots, G_k$ $(k \geq 2)$ on $n_1, n_2, \cdots, n_k$ vertices, respectively, and $n = \sum\limits_{i=1}^{k} n_i - k + 1$. Then the distance matrix $D(G)$ of G is unimodular equivalent to the following matrix*

$$I_{n-k-1} \oplus \begin{pmatrix} k & 1 & 1 & \cdots & 1 & 1 \\ 1 & n_1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & n_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & n_{k-1} & 0 \\ 1 & 0 & 0 & \cdots & 0 & n_k \end{pmatrix}.$$

Hence, the Smith normal form of $D(G)$ does not depend on the interconnection of the cliques.

Proof Since G is a block graph with at least two blocks, it must have a leaf block (that is, the block contains only one cut vertex of G). Without loss of generalization, suppose that G_k is a leaf block of G and the vertex c is the unique cut vertex of G in G_k. Let D_{k-1} be the distance matrix of the graph $G - (G_k - c)$. Then the distance matrix $D_k = D(G)$ of G can be written as follows:

$$D_k = \begin{pmatrix} D_{k-1} & \alpha + j & \alpha + j & \cdots & \alpha + j \\ (\alpha + j)^{\mathrm{T}} & 0 & 1 & \cdots & 1 \\ (\alpha + j)^{\mathrm{T}} & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (\alpha + j)^{\mathrm{T}} & 1 & 1 & \cdots & 0 \end{pmatrix},$$

where α is the column of D_{k-1} corresponding to the vertex c. Note that $\alpha(c) = d(c, c) = 0$. Subtracting the row and column corresponding to the vertex c from the rows and columns of $G_k - c$, we get

$$D(G) \sim \begin{pmatrix} D_{k-1} & j & j & \cdots & j \\ j^{\mathrm{T}} & -2 & -1 & \cdots & -1 \\ j^{\mathrm{T}} & -1 & -2 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ j^{\mathrm{T}} & -1 & -1 & \cdots & -2 \end{pmatrix} = \begin{pmatrix} D_{k-1} & J \\ J & -I - J \end{pmatrix}.$$

Let $P = \begin{pmatrix} I & 0 \\ 0 & P_{n_k - 1} \end{pmatrix}$, $Q = \begin{pmatrix} I & 0 \\ 0 & Q_{n_k - 1} \end{pmatrix}$ be two $n \times n$ matrices, where $P_{n_k - 1}$ and $Q_{n_k - 1}$ are the matrices defined as in (2.2). Then P, Q are unimodular and

$$P \begin{pmatrix} D_{k-1} & J \\ J & -I - J \end{pmatrix} Q = \begin{pmatrix} D_{k-1} & JQ_{n_k - 1} \\ P_{n_k - 1} J & P_{n_k - 1}(-I - J) Q_{n_k - 1} \end{pmatrix}.$$

Since

$$P_{n_k-1}J=\begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \\ n_k-1 & n_k-1 & \cdots & n_k-1 \end{pmatrix}, \quad JQ_{n_k-1}=\begin{pmatrix} 0 & \cdots & 0 & -1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -1 \\ 0 & \cdots & 0 & -1 \\ 0 & \cdots & 0 & -1 \end{pmatrix},$$

$$P_{n_k-1}(-I-J)Q_{n_k-1}=-I_{n_k-3}\oplus\begin{pmatrix} -1 & -1 \\ 0 & -n_k \end{pmatrix}.$$

Thus

$$D\sim\left(\begin{array}{ccc|ccccc} & & & 0 & \cdots & 0 & 0 & -1 \\ & & & \vdots & \ddots & \vdots & \vdots & \vdots \\ & D_{k-1} & & 0 & \cdots & 0 & 0 & -1 \\ & & & -1 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \vdots & 0 & \cdots & -1 & 0 & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & -1 & -1 \\ 1 & \cdots & 1 & & & & & \\ n_k-1 & \cdots & n_k-1 & 0 & \cdots & 0 & 0 & -n_k \end{array}\right).$$

Subtracting the $(n-1)$-th column from the n-th column and adding the $(n-1)$-th column to the columns $1,2,\cdots,n-n_k+1$, we have

$$D\sim I_{n_k-2}\oplus\begin{pmatrix} D_{k-1} & j \\ (n_k-1)j^{\mathrm{T}} & -n_k \end{pmatrix}.$$

For D_{k-1}, continuing the above reduced way, we finally obtain

$$D\sim I_{n_k-2}\oplus\cdots\oplus I_{n_2-2}\oplus\begin{pmatrix} 0 & 1 & \cdots & 1 & 1 & -1 & \cdots & -1 \\ 1 & 0 & \cdots & 1 & 1 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 0 & -1 & \cdots & -1 \\ n_2-1 & n_2-1 & \cdots & n_2-1 & n_2-1 & -n_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ n_k-1 & n_k-1 & \cdots & n_k-1 & n_k-1 & 0 & \cdots & -n_k \end{pmatrix}.$$

For the above matrix, subtracting the first row and the first column from the rows and columns $2,\cdots,n_1$, we get

$$D(G) \sim I_{n-n_1+1} \oplus \begin{pmatrix} 0 & 1 & \cdots & 1 & 1 & -1 & \cdots & -1 \\ 1 & -2 & \cdots & -1 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & -1 & \cdots & -1 & -2 & 0 & \cdots & 0 \\ n_2-1 & 0 & \cdots & 0 & 0 & -n_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ n_k-1 & 0 & \cdots & 0 & 0 & 0 & \cdots & -n_k \end{pmatrix}$$
$$\sim \begin{pmatrix} -(k-1) & j^{\mathrm{T}} & -j^{\mathrm{T}} \\ j & -I-J & 0 \\ -j & 0 & N \end{pmatrix},$$

where $N = \mathrm{diag}(-n_2, -n_3, \cdots, -n_k)$.

By some computation, we have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & P_{n_1-1} & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} -(k-1) & j^{\mathrm{T}} & -j^{\mathrm{T}} \\ j & -I-J & 0 \\ -j & 0 & N \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & Q_{n_1-1} & 0 \\ 0 & 0 & I \end{pmatrix}$$
$$= \begin{pmatrix} -(k-1) & j^{\mathrm{T}} Q_{n_1-1} & -j^{\mathrm{T}} Q_{n_1-1} \\ P_{n_1-1} j & P_{n_1-1}(-I-J) Q_{n_1-1} & 0 \\ -j & 0 & N \end{pmatrix}$$
$$= -I_{n_1-3} \oplus \begin{pmatrix} -(k-1) & 0 & -1 & -1 & \cdots & -1 \\ 1 & -1 & -1 & 0 & \cdots & 0 \\ n_1-1 & 0 & -n_1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & -n_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & 0 & \cdots & -n_k \end{pmatrix}$$
$$\sim I_{n_1-2} \oplus \begin{pmatrix} -k & -1 & -1 & \cdots & -1 \\ -1 & -n_1 & 0 & \cdots & 0 \\ -1 & 0 & -n_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & -n_k \end{pmatrix}$$
$$\sim I_{n_1-2} \oplus \begin{pmatrix} k & 1 & 1 & \cdots & 1 \\ 1 & n_1 & 0 & \cdots & 0 \\ 1 & 0 & n_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & n_k \end{pmatrix}.$$

Note that $\sum\limits_{i=1}^{k}(n_i-2) = \sum\limits_{i=1}^{k} n_i - 2k = n-k-1$. Hence the proof is completed.

Although we can not give explicit formula of the Smith normal form of the $(k+1)\times(k+1)$-matrix in Theorem 2.1 in general case, the next lemma is useful for the computation of its Smith normal form.

Lemma 2.2 *Let $A(c_0; b_1, b_2, \cdots, b_n; a_1, a_2, \cdots, a_n)$ be an $(n+1)$-by-$(n+1)$ integer matrix as follows:*

$$A(c_0; b_1, b_2, \cdots, b_n; a_1, a_2, \cdots, a_n) = \begin{pmatrix} c_0 & b_1 & b_2 & \cdots & b_n \\ b_1 & a_1 & 0 & \cdots & 0 \\ b_2 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n & 0 & 0 & \cdots & a_n \end{pmatrix},$$

where $a_1, a_2, \cdots, a_n$ are nonzero integers. For $2 \le r \le n$, let U and V be nonempty subsets of $\{1, 2, \cdots, n, n+1\}$ with $|U| = |V| = r$, and $A[U|V]$ be the $r \times r$-submatrix of A with rows in U and columns in V. Then:

(1) *If $1 \notin U$ and $1 \notin V$, then* $\det A[U|V]$ *is $\prod_{j=1}^{r} a_{i_j}$ when $U = V = \{i_1, i_2, \cdots, i_r\}$, 0 else.*

(2) *If 1 is in exactly one of U and V, then* $\det A[U|V]$ *is $b_{i_r} \prod_{j=1}^{r-1} a_{i_j}$ when $U - \{1\} \subset V = \{i_1, i_2, \cdots, i_r\} \not\ni 1$ or $V - \{1\} \subset U = \{i_1, i_2, \cdots, i_r\} \not\ni 1$, 0 else.*

(3) *If $1 \in U$ and $1 \in V$, then*

$$\det A[U|V] = \begin{cases} \Big(c_0 - \sum_{j=1}^{r-1} \dfrac{b_{i_j}}{a_{i_j}}\Big) \prod_{j=1}^{r-1} a_{i_j}, & \text{if } U = V = \{1, i_1, \cdots, i_{r-1}\}, \\ b_\alpha b_\beta \prod_{j=1}^{r-2} a_{i_j}, & \text{if } U = \{1, i_1, \cdots, i_{r-2}, \alpha\} \text{ and } \\ & \quad V = \{1, i_1, \cdots, i_{r-2}, \beta\}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof In case (1), considering all possible determinants of $r \times r$-submatrices, they clearly are either 0 or of the form $\prod_{j=1}^{r} a_{i_j}$.

In case (2), without loss of generalization, assume that $1 \in U$ and $1 \notin V = \{i_1, i_2, \cdots, i_r\}$. In order that the other $r-1$ rows are chosen from $\{2, 3, \cdots, n, n+1\}$ to yield an $r \times r$-submatrix with nonzero determinant, we have $U - \{1\} \subset V$. If U contains $x \neq 1$ which is not in V, then row x will be all zeros in the $A[U|V]$ and $\det A[U|V] = 0$. If indeed $U - \{1\} \subset V$, then $A[U|V]$ is of the form

$$\begin{pmatrix} b_{i_1} & b_{i_2} & \cdots & b_{i_{r-1}} & b_{i_r} \\ a_{i_1} & 0 & \cdots & 0 & 0 \\ 0 & a_{i_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{i_{r-1}} & 0 \end{pmatrix},$$

and $\det A[U|V] = b_{i_r} \prod_{j=1}^{r-1} a_{i_j}$ holds.

In case (3), there are three subcases, depending upon how U and V intersect. If the symmetric difference $(U-V)\cup(V-U)$ contains more than two elements, then one can check that $A[U|V]$ must contain a zero row or a zero column and $\det A[U|V]=0$.

If $(U-V)\cup(V-U)=\{\alpha,\beta\}$, then one may assume that $U=\{1,i_1,\cdots,i_{r-2},\alpha\}$ and $V=\{1,i_1,,\cdots,i_{r-2},\beta\}$. Thus $A[U|V]$ is of the form

$$\begin{pmatrix} c_0 & b_{i_1} & b_{i_2} & \cdots & b_{i_{r-2}} & b_\beta \\ b_{i_1} & a_{i_1} & 0 & \cdots & 0 & 0 \\ b_{i_2} & 0 & a_{i_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{i_{r-2}} & 0 & 0 & \cdots & a_{i_{r-2}} & 0 \\ b_\alpha & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

and $\det A[U|V] = b_\alpha b_\beta \prod_{j=1}^{r-2} a_{i_j}$.

If $U=V=\{1,i_1,i_2,\cdots,i_{r-1}\}$, then it is easy to check

$$\det A[U|V] = \left(c_0 - \sum_{j=1}^{r-1} \frac{b_{i_j}}{a_{i_j}}\right) \prod_{j=1}^{r-1} a_{i_j}.$$

The proof is completed.

Using Lemma 2.2, it is not difficult to obtain:

Corollary 2.1 *Let $A=A(c;b,\cdots,b;a,\cdots,a)$ be a matrix of order $n+1$ defined in Lemma* 2.2. *Then*

$$Snf(A(c;b,\cdots,b;a,\cdots,a)) = \mathrm{diag}(s_1,s_2,a,\cdots,a,s_{n+1}),$$

where $s_1=gcd(c,a,b)$, $s_2=\frac{gcd(a^2,b^2,ca,ba)}{gcd(c,b,a)}$, $s_{n+1}=\frac{a(ac-nb^2)}{gcd(c,b,a)}$.

If all cliques of a block graph G have the same size, then the Smith normal form of $D(G)$ has a simple form as follows.

Corollary 2.2 *Let G be a block graph with k cliques K_p and $n=pk-k+1$ vertices. Then*

$$Snf(D(G)) = I_{n-k+1} \oplus pI_{k-2} \oplus (p(n-1)).$$

Proof By Theorem 2.1, $D(G)$ is unimodular equivalent to $I_{n-k-1} \oplus A(k;1,\cdots,1;$

$p, \cdots, p)$. Applying Corollary 2.1, the Smith normal form of $A(k; 1, \cdots, 1; p, \cdots, p)$ is $I_2 \oplus pI_{k-2} \oplus (p(pk-k))$. Since $n = pk - k + 1$, hence Corollary 2.2 holds.

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