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A NEW CLASS OF SOLUTIONS OF VACUUM EINSTEIN'S FIELD EQUATIONS WITH COSMOLOGICAL CONSTANT*†

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Abstract

In this paper, a new class of solutions of the vacuum Einstein's field equations with cosmological constant is obtained. This class of solutions possesses the naked physical singularity. The norm of the Riemann curvature tensor of this class of solutions takes infinity at some points and the solutions do not have any event horizon around the singularity.

Keywords vacuum Einstein's field equations; cosmological constant; exact solutions; naked physical singularity

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1 Introduction

The cosmological constant problem is one of the most outstanding and unsolved problems in cosmology. It has been a focal point of interest [1-4]. From our point of view, the universe possesses a non-zero cosmological constant which is considered as the vacuum energy density. The cosmological term provides a repulsive force opposing the gravitational pull between the galaxies. The recent measurements of type Ia Supernovae observations [5,6] and findings from the anisotropy measurements of the cosmic microwave background by the Wilkinson Microwave Anisotropy Probe [7] have shown that our universe is undergoing an accelerated expansion. It is the most accepted idea that a mysterious dominant component dubbed dark energy leads to this cosmic acceleration. The cosmological constant is a strong candidate for dark energy which pushes the universe accelerated expansion.

Exact solutions of Einstein's field equations with cosmological constant, which offer an alternative and complementary approach to study cosmological models, have been investigated from time to time. As early as 1918, Kottler extended the

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Schwarzschild solution and obtained the static spherically symmetric exterior solution with cosmological constant. Kramer et al. got the Reissner-Nordsröm exterior solution with cosmological constant [8]. Xu, Wu and Huang extended the Florides' solution to the case with cosmological constant [9]. Recently, Nurowski provided the first examples of vacuum metrics with cosmological constant which have a twisting quadruple principal null direction [10]. Kamenshchik and Mingarelli found an exact solution of a Bianchi-I Universe in the presence of dust, stiff matter and a negative cosmological constant, generalising the well-known Heckmann-Schucking solution [11]. Landry, Abdelqader and Lake studied the McVittie solution with a negative cosmological constant and they found that cosmological constant $\Lambda < 0$ ensures collapse to a Big Crunch [12]. Zubairi and Weber [13] derived the modified Tolman-Oppenheimer-Volkoff equations which account for a finite value of the cosmological constant for spherically symmetric mass distributions.

In this paper, we consider the vacuum Einstein's field equations with cosmological constant of the following form

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0, \tag{1}$$

or equivalently

$$R_{\mu\nu} = \Lambda g_{\mu\nu}. \tag{2}$$

They are solved by presenting a special form of Lorentzian metric and taking a proper ansatz. It is shown that this new class of solutions possesses naked physical singularity. Moreover, the solutions presented in this paper extend the results of [14] to the case with cosmological constant.

2 The Solutions with Cosmological Constant

In the coordinate (t, x, y, z), consider the metric of the form

$$\mathrm{d}s^2 = A\mathrm{d}t^2 - \frac{1}{A}\mathrm{d}x^2 - B\mathrm{d}y^2 - y^2 B\mathrm{d}z^2, \tag{3}$$

where $A = A(t, x)$ and $B = B(t, x)$. It is easy to verify that the determinant of metric $(g_{\mu\nu})$ is given by

$$g \stackrel{\triangle}{=} \det(g_{\mu\nu}) = -B^2 y^2 < 0,$$

so the metric $(g_{\mu\nu})$ is Lorentzian.

In view of metric (3), the vacuum Einstein's field equations with cosmological constant (2) reduce to the following system

$$B^2A^3A_{xx} - 2B^2A_t^2 + B^2AA_{tt} - 2A^2BB_{tt} + ABA_tB_t$$
$$+A^3BA_xB_x + A^2B_t^2 = 2\Lambda A^3B^2, \tag{4}$$
$$2ABB_{xt} - BB_tA_x + BA_tB_x - AB_tB_x = 0, \tag{5}$$
$$B^2A^3A_{xx} - 2B^2A_t^2 + B^2AA_{tt} + 2A^4BB_{xx} + ABA_tB_t$$
$$+A^3BA_xB_x - A^4B_x^2 = 2\Lambda A^3B^2, \tag{6}$$
$$A_tB_t + A^2A_xB_x + A^3B_{xx} - AB_{tt} = 2\Lambda A^2B. \tag{7}$$

In order to simplify system (4)-(7), we take the ansatz as follows

$$A = A(x-t) \tag{8}$$

and

$$B = B(x-t). \tag{9}$$

Define $r = x - t$ and denote $' \equiv \frac{\mathrm{d}}{\mathrm{d}r}$, then equation (5) reduces to

$$2BB'' = B'^2, \tag{10}$$

which gives

$$B = \frac{m^2r^2}{4} + \frac{mkr}{2} + \frac{k^2}{4}. \tag{11}$$

Here m and k are two constants of integration. Letting

$$m = 1 \tag{12}$$

and

$$k = 0, \tag{13}$$

we have

$$B = \frac{r^2}{4}. \tag{14}$$

Substituting (14) into equations (4), (6) and (7) yeilds

$$A''(A^3 + A) - 2A'^2 + \frac{2A'(A + A^3)}{r} = 2\Lambda A^3 \tag{15}$$

and

$$A'(A^2+1) + \frac{A^3 - A}{r} = \Lambda A^2 r. \tag{16}$$

Solving equation (15) gives

$$A = \frac{\Lambda r^3 - 3c_1r + 3c_2 \pm \sqrt{\Lambda^2r^6 - 6\Lambda c_1r^4 + 6\Lambda c_2r^3 + 9c_1^2r^2 - 18c_1c_2r + 9c_2^2 + 36r^2}}{6r}, \tag{17}$$

where c_1 and c_2 are integration constants. Taking $c_1 = 0$ yields

$$A = \frac{\Lambda r^3 + 3c_2 \pm \sqrt{\Lambda^2 r^6 + 6\Lambda c_2 r^3 + 9c_2^2 + 36r^2}}{6r}, \tag{18}$$

which are exactly the solutions of equation (16). In view of $r = x - t$, we finally have

$$A = \frac{\Lambda(x-t)^3 + 3c_2 \pm \sqrt{\Lambda^2(x-t)^6 + 6\Lambda c_2(x-t)^3 + 9c_2^2 + 36(x-t)^2}}{6(x-t)} \tag{19}$$

and

$$B = \frac{(x-t)^2}{4}. \tag{20}$$

From the above discussion, we obtain the following theorem.

Theorem 2.1 *In the coordinates (t, x, y, z), the vacuum Einstein's filed equations with cosmological constant Λ have the following solutions*

$$\begin{aligned} \mathrm{d}s^2 = & \frac{\Lambda(x-t)^3 + 3c_2 \pm \sqrt{\Lambda^2(x-t)^6 + 6\Lambda c_2(x-t)^3 + 9c_2^2 + 36(x-t)^2}}{6(x-t)}\mathrm{d}t^2 \\ & - \frac{6(x-t)}{\Lambda(x-t)^3 + 3c_2 \pm \sqrt{\Lambda^2(x-t)^6 + 6\Lambda c_2(x-t)^3 + 9c_2^2 + 36(x-t)^2}}\mathrm{d}x^2 \\ & - \frac{(x-t)^2}{4}\mathrm{d}y^2 - \frac{y^2(x-t)^2}{4}\mathrm{d}z^2, \end{aligned} \tag{21}$$

where c_2 is an integration constant.

As is shown by the definition

$$r = x - t, \tag{22}$$

we get the following theorem.

Theorem 2.2 *In the coordinates (t, r, y, z), the solutions of vacuum Einstein's filed equations with cosmological constant $R_{\mu\nu} = \Lambda g_{\mu\nu}$ $(\mu, \nu = 0, 1, 2, 3)$ become*

$$\begin{aligned} \mathrm{d}s^2 = & \frac{\Lambda r^3 + 3c_2}{3r}\mathrm{d}t^2 + \frac{\Lambda r^3 + 3c_2 \mp \sqrt{\Lambda^2 r^6 + 6\Lambda c_2 r^3 + 9c_2^2 + 36r^2}}{3r}\mathrm{d}t\mathrm{d}r \\ & + \frac{\Lambda r^3 + 3c_2 \mp \sqrt{\Lambda^2 r^6 + 6\Lambda c_2 r^3 + 9c_2^2 + 36r^2}}{6r}\mathrm{d}r^2 - \frac{r^2}{4}\mathrm{d}y^2 - \frac{y^2 r^2}{4}\mathrm{d}z^2. \end{aligned} \tag{23}$$

3 Singularity

This section is devoted to the analysis of singularity of (23) of the vacuum Einstein's field equations with cosmological constant. By calculations, the corresponding Riemann curvature tensor of (23) reads as

$$R_{0101} = -\frac{\Lambda r^3 + 3c_2}{3r^3}, \tag{24}$$

$$R_{2323} = \frac{\Lambda r^4 y^2}{48} + \frac{c_2 r y^2}{16}, \tag{25}$$

$$R_{0202} = \frac{(\Lambda r^3 + 3c_2)(-2\Lambda r^3 + 3c_2)}{72r^2}, \tag{26}$$

$$R_{0303} = \frac{y^2(\Lambda r^3 + 3c_2)(-2\Lambda r^3 + 3c_2)}{72r^2}, \tag{27}$$

$$R_{0212} = \frac{(\Lambda r^3 + 3c_2 \mp \sqrt{\Lambda^2 r^6 + 6\Lambda c_2 r^3 + 9c_2^2 + 36r^2})(-2\Lambda r^3 + 3c_2)}{144r^2}, \tag{28}$$

$$R_{0313} = \frac{y^2(\Lambda r^3 + 3c_2 \mp \sqrt{\Lambda^2 r^6 + 6\Lambda c_2 r^3 + 9c_2^2 + 36r^2})(-2\Lambda r^3 + 3c_2)}{144r^2}, \tag{29}$$

$$R_{1212} = \frac{\mp(-2\Lambda r^3 + 3c_2)\sqrt{\Lambda^2 r^6 + 6\Lambda c_2 r^3 + 9c_2^2 + 36r^2} - 2\Lambda^2 r^6 - 3\Lambda c_2 r^3 + 9c_2^2}{144r^2}, \tag{30}$$

$$R_{1313} = \frac{y^2[\mp(-2\Lambda r^3 + 3c_2)\sqrt{\Lambda^2 r^6 + 6\Lambda c_2 r^3 + 9c_2^2 + 36r^2} - 2\Lambda^2 r^6 - 3\Lambda c_2 r^3 + 9c_2^2]}{144r^2} \tag{31}$$

and the other $R_{\alpha\beta\mu\nu} = 0$.

Obviously, the non-vanishing Riemann curvature tensor tends to infinity when $r \to 0$. By further calculations and analysis, we obtain the norm of the Riemann curvature tensor

$$\mathbf{R} \overset{\triangle}{=} R^{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu} \longrightarrow \infty, \tag{32}$$

while $r \to 0$. Thus, $r = 0$ is a physical singularity of the space-time described by (23). From (23)-(31), it is easy to get that no event horizon exists around the physical singularity $r = 0$. It can be concluded that (23) possess a naked physical singularity $r = 0$.

4 Summary and Discussions

When $\Lambda = 0$, (21) reduce to

$$\begin{aligned} ds^2 = &\frac{c_2 \pm \sqrt{c_2^2 + 4(x-t)^2}}{2(x-t)} dt^2 - \frac{2(x-t)}{c_2 \pm \sqrt{c_2^2 + 4(x-t)^2}} dx^2 \\ &- \frac{(x-t)^2}{4} dy^2 - \frac{y^2(x-t)^2}{4} dz^2, \end{aligned} \tag{33}$$

which are the solutions of the vacuum Einstein's field equations with vanishing cosmological constant

$$R_{\mu\nu} = 0.$$

In particular, taking $A=\frac{c_2+\sqrt{4(x-t)^2+c_2^2}}{2(x-t)}$ and $c_2=1$, we have

$$ds^2=\frac{1+\sqrt{1+4(x-t)^2}}{2(x-t)}dt^2-\frac{2(x-t)}{1+\sqrt{1+4(x-t)^2}}dx^2-\frac{(x-t)^2}{4}dy^2-\frac{y^2(x-t)^2}{4}dz^2, \quad (34)$$

which is exactly the solution presented in [14]. Thus the solutions in Theorem 2.1 include the results obtained in [14] and extend the solutions to the case with cosmological constant.

Moreover, the solutions with cosmological constant obtained here possess naked physical singularity, that is, no event horizon exists around the physical singularity. It is expected that this new class of solutions can be applied to modern cosmology.

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