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GLOBAL ATTRACTIVITY IN AN ALMOST PERIODIC PREDATOR-PREY-MUTUALIST SYSTEM*†

Liya Yang, Fengde Chen‡, Chengqiang Wu
(*College of Math. and Computer Science, Fuzhou University, Fujian 350116, PR China*)
Xiangdong Xie
(*Dept. of Math., Ningde Normal University, Fujian 352300, PR China*)

Abstract

In this paper, the almost periodic predator-prey-mutualist model with Holling type II functional response is discussed. A set of sufficient conditions which guarantee the uniform persistence and the global attractivity of the system are obtained. For the almost periodic case, by constructing a suitable Lyapunov function, sufficient conditions which guarantee the existence of a unique globally attractive positive almost periodic solution of the system are obtained. An example together with its numerical simulations shows the feasibility of the main results.

Keywords almost periodic solution; predator-prey-mutualist system; functional response; Lyapunov function; global attractivity

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1 Introduction

As was pointed out by Berryman [1], the dynamic relationship between predator and prey has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance. Already, the predator-prey model has been studied by several scholars [2-10]. For example, Das etc. [8] investigated a three species ecosystem consisting of a prey, predator and a top predator. They derived the criteria for local and global stability of all the eight equilibrium points using Routh-Hurwitz and Lyapunov function. Wu and Li [9] studied the permanence and global attractivity of the discrete predator-

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‡Corresponding author. E-mail: fdchen@263.net

prey system with Hassell-Varley-Holling III type functional response. Chen and Chen [10] proposed a ratio-dependent predator-prey model incorporating a prey refuge. They studied the global stability, limit cycle and Hopf bifurcation of the system.

Though mutualism is one of the most important relationships in the real world, for instance, ants prevent herbivores from feeding on plants (see [11]) and ants prevent predators from feeding on aphids (see [12,13]). As was pointed out by Murray [14]: "this area has not been as widely studied as the others even though its importance is comparable to that of predator-prey and competition interactions." To this end, Rai and Krawcewicz [15] proposed the following three species predator-prey-mutualist system:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \alpha x\Big(1 - \dfrac{x}{K}\Big) - \dfrac{\beta x z}{1+my}, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = \gamma y\Big(1 - \dfrac{y}{lx + L_0}\Big), \\ \dfrac{\mathrm{d}z}{\mathrm{d}t} = z\Big(-s + \dfrac{c\beta x}{1+my}\Big), \end{cases} \tag{1.1}$$

where $x(t)$, $y(t)$ and $z(t)$ denote the densities of prey, mutualist and predator population at any time t, respectively. They applied the equivariant degree method to study Hopf bifurcations phenomenon of the system.

Recently, Yang, Xie and Wu [16] argued that due to seasonal effects of weather, temperature, food supply, mating habits etc, a more appropriate system should be the non-autonomous case, and they proposed and studied the following system:

$$\begin{cases} \dot{x} = x\Big(a_1(t) - b_1(t)x - \dfrac{c_1(t)z}{d_1(t) + d_2(t)y}\Big), \\ \dot{y} = y\Big(a_2(t) - \dfrac{y}{d_3(t) + d_4(t)x}\Big), \\ \dot{z} = z\Big(-a_3(t) + \dfrac{k_1(t)c_1(t)x}{d_1(t) + d_2(t)y} - b_2(t)z\Big). \end{cases} \tag{1.2}$$

By using the Brouwer fixed pointed theorem and constructing a suitable Lyapunov function, the authors obtained a set of sufficient conditions for the existence of a globally asymptotically stable periodic solution of system (1.2).

It brings to our attention that in systems (1.1) and (1.2), the authors did not consider the functional response of the predator species, which motivates us to study a suitable predator-prey system incorporating some functional response of the predator species, and to propose the following three species predator-prey-mutualist system:

$$\begin{cases} \dot{x} = x\Big(a_1(t) - b_1(t)x - \dfrac{c_1(t)z}{d_1(t) + x + d_2(t)y}\Big), \\ \dot{y} = y\Big(a_2(t) - \dfrac{y}{d_3(t) + d_4(t)x}\Big), \\ \dot{z} = z\Big(- a_3(t) - b_2(t)z + \dfrac{c_2(t)x}{d_1(t) + x + d_2(t)y}\Big), \end{cases} \tag{1.3}$$

where x is the density of the prey, y is the density of the mutualist and z is the density of the predator. The functions $a_i(t)$ $(i = 1, 2, 3)$, $b_1(t)$, $b_2(t)$, $c_1(t)$, $c_2(t)$, $d_j(t)$ $(j = 1, 2, 3, 4)$ are continuous defined on $[0, +\infty)$; $a_i(t)$ $(i = 1, 2, 3)$, $b_1(t)$, $b_2(t)$ are strictly positive, $a_1(t)$ is the intrinsic growth rate of prey specie x, $a_2(t)$ is the intrinsic growth rate of mutualist y, $a_3(t)$ is the death rate of the predator specie z. The functions d_4 and d_2 reflect the mutualist effect.

To the best of the authors knowledge, to this day, still no scholars investigate the almost periodic solution of system (1.3), it is well known that the assumption of almost periodicity of the coefficients of (1.3) is a way of incorporating the time dependent variability of the environment, especially when the various components of the environment are periodic with not necessary commensurate periods (e.g. seasonal effects of weather, food supplies, mating habits, harvesting etc.) [17-20]. We arrange the rest of this paper as follows: In Section 2, by the differential inequality theory, sufficient conditions which guarantee the uniform persistence of system (1.3) are obtained; after that, by constructing a suitable Lyapunov function, some sufficient conditions which ensure the global attractivity of system (1.3) are obtained. In Section 3, a criterion is established for the existence of a unique globally attractive positive almost periodic solution of system (1.3). In Section 4, a suitable example together with its numeric simulations is given to illustrate the main results of this paper. We end this paper by a briefly discussion.

Throughout this paper, we shall use the following notations:

$$f^l = \inf_{t\in R} f(t), \quad f^u = \sup_{t\in R} f(t).$$

2 General Nonautonomous Case

Lemma 2.1 $R^3_+ = \big\{(x, y, z) |\, x\geq 0, y\geq 0, z\geq 0\big\}$ *is invariant with respect to* (1.3).

Proof Since

$$\begin{cases} x(t) = x(0) \exp \displaystyle\int_0^t \Big(a_1(s) - b_1(s)x(s) - \frac{c_1(s)z(s)}{d_1(s) + x(s) + d_2(s)y(s)}\Big)\mathrm{d}s, \\ y(t) = y(0) \exp \displaystyle\int_0^t \Big(a_2(s) - \frac{y(s)}{d_3(s) + d_4(s)x(s)}\Big)\mathrm{d}s, \\ z(t) = z(0) \exp \displaystyle\int_0^t \Big(- a_3(s) + \frac{c_2(s)x(s)}{d_1(s) + x(s) + d_2(s)y(s)} - b_2(s)z(s)\Big)\mathrm{d}s, \end{cases} \tag{2.1}$$

the assertion of the lemma follows immediately for all $t \in [0, +\infty)$.

It follows from Lemma 2.1 that any solution of (1.3) with a nonnegative initial condition remains nonnegative.

Lemma 2.2 *If $m_1 > 0$ and $m_3 > 0$, then the set S defined by*

$$S = \{w = (x, y, z) \in R_+^3 |\, m_1 \le x \le M_1, m_2 \le y \le M_2, m_3 \le z \le M_3\}$$

is invariant with respect to (1.3), *where m_i and M_i $(i = 1, 2, 3)$ will be defined below.*

Proof From the first equation of system (1.3), we obtain $\dot{x} \le x(a_1^u - b_1^l x)$. If

$$0 < x(0) \le \frac{a_1^u}{b_1^l} := M_1$$

holds, then we have

$$x(t) \le M_1, \quad t \ge 0. \tag{2.2}$$

From the second equation of (1.3), it follows that

$$\dot{y} \le y\Big(a_2^u - \frac{y}{d_3^u + d_4^u M_1}\Big).$$

This together with $0 < y(0) \le a_2^u(d_3^u + d_4^u M_1) := M_2$ implies

$$y(t) \le M_2, \quad t \ge 0. \tag{2.3}$$

From the third equation of (1.3), it follows that

$$\dot{z} \le z\Big(-a_3^l + \frac{c_2^u M_1}{d_1^l} - b_2^l z\Big).$$

If

$$0 < z(0) \le \frac{c_2^u M_1 - a_3^l d_1^l}{d_1^l b_2^l} := M_3$$

holds, then

$$z(t) \le M_3, \quad t \ge 0. \tag{2.4}$$

From the second equation of system (1.3), one has

$$\dot{y} \ge y\Big(a_2^l - \frac{y}{d_3^l}\Big),$$

which implies that if $y(0) \ge a_2^l d_3^l := m_2$ holds, then

$$y(t) \ge m_2, \quad t \ge 0. \tag{2.5}$$

(2.5) combining with the first equation of system (1.3) leads to

$$\dot{x} \ge x\Big(a_1^l - b_1^u x - \frac{c_1^u M_3}{d_1^l + d_2^l m_2}\Big).$$

It implies that if

$$x(0) \geq \frac{a_1^l - \dfrac{c_1^u M_3}{d_1^l + d_2^l m_2}}{b_1^u} = \frac{a_1^l(d_1^l + d_2^l m_2) - c_1^u M_3}{(d_1^l + d_2^l m_2) b_1^u} := m_1$$

holds, then

$$x(t) \geq m_1, \quad t \geq 0. \tag{2.6}$$

From (1.3), (2.3) and (2.6), we have

$$\dot{z} \geq z\Big(- a_3^u + \frac{c_2^l m_1}{d_1^u + d_2^u M_2 + M_1} - b_2^u z\Big).$$

If

$$z(0) \geq \frac{c_2^l m_1 - a_3^u(d_1^u + d_2^u M_2 + M_1)}{(d_1^u + d_2^u M_2 + M_1) b_2^u} := m_3$$

holds, then

$$z(t) \geq m_3, \quad t \geq 0. \tag{2.7}$$

The above analysis shows that

$$0 < m_1 \leq x(t) \leq M_1, \quad 0 < m_2 \leq y(t) \leq M_2, \quad 0 < m_3 \leq z(t) \leq M_3, \quad t \geq 0.$$

This completes the proof of Lemma 2.2.

With a slightly modification of the proof of Lemma 2.2, we could also obtain following result.

Lemma 2.3 *Assume that $m_1 > 0$ and $m_3 > 0$. Let $F(t) = (x(t), y(t), z(t))$ be any positive solution of system* (1.3)*, then we have*

$$\begin{aligned} m_1 &\leq \liminf_{t\to\infty} x(t) \leq \limsup_{t\to\infty} x(t) \leq M_1, \\ m_2 &\leq \liminf_{t\to\infty} y(t) \leq \limsup_{t\to\infty} y(t) \leq M_2, \\ m_3 &\leq \liminf_{t\to\infty} z(t) \leq \limsup_{t\to\infty} z(t) \leq M_3. \end{aligned}$$

As a direct corollary of Lemma 2.3, we have:

Theorem 2.1 *Under the assumptions $m_1 > 0$ and $m_3 > 0$, system* (1.3) *is uniformly persistent.*

Theorem 2.2 *If the coefficients of system* (1.3) *satisfy the following conditions:*

(I) $m_1 > 0,\ m_3 > 0$;

(II) *there exist positive constants μ_1, μ_2, μ_3 and δ such that*

$$\min_{t\in R}\{\varphi(t), \psi(t), \phi(t)\} > \delta,$$

where

$$\varphi(t) = \mu_1 b_1(t) - \frac{\mu_1 c_1(t) M_3}{d_1^2(t)} - \frac{\mu_2 d_4(t) M_2}{d_3^2(t)} - \frac{\mu_3 c_2(t)}{d_1(t)}; \quad \phi(t) = \mu_3 b_2(t) - \frac{\mu_1 c_1(t)}{d_1(t)};$$

$$\psi(t) = \frac{\mu_2}{d_3(t)+d_4(t)M_1} - \frac{\mu_1 c_1(t) d_2(t) M_3}{d_1^2(t)} - \frac{\mu_3 c_2(t) d_2(t) M_1}{d_1^2(t)}.$$

Then system (1.3) *is globally attractive.*

Proof Let $W(t) = (x(t), y(t), z(t))$ and $F(t) = (x^*(t), y^*(t), z^*(t))$ be any two positive solutions of (1.3).

Set $V(t) = V_1(t) + V_2(t) + V_3(t)$, where

$$V_1(t) = \mu_1 |\ln x(t) - \ln x^*(t)|, \quad V_2(t) = \mu_2 |\ln y(t) - \ln y^*(t)|, \quad V_3(t) = \mu_3 |\ln z(t) - \ln z^*(t)|.$$

By simple computation, one has

$$\begin{aligned}
D^+V_1(t) &\le -\mu_1\Big[b_1(t) - \frac{c_1(t)z(t)}{d_1^2(t)}\Big]|x^*(t)-x(t)| + \frac{\mu_1 c_1(t) d_2(t) z(t)}{d_1^2(t)}|y^*(t)-y(t)| \\
&\quad + \frac{\mu_1 c_1(t)}{d_1(t)}|z^*(t)-z(t)|, \\
D^+V_2(t) &\le \frac{\mu_2 d_4(t) y(t)}{d_3^2(t)}|x^*(t)-x(t)| - \frac{\mu_2}{d_3(t)+d_4(t)x^*(t)}|y^*(t)-y(t)|, \\
D^+V_3(t) &\le \frac{c_2(t)\mu_3}{d_1(t)}|x^*(t)-x(t)| + \frac{\mu_3 c_2(t) d_2(t) x(t)}{d_1^2(t)}|y^*(t)-y(t)| - \mu_3 b_2(t)|z^*(t)-z(t)|.
\end{aligned}$$

From Lemma 2.3 it follows that there exist a $T_1 > 0$ large enough and an ε small enough such that for all $t > T_1$, one has

$$\begin{aligned}
m_1 - \varepsilon &\le x(t), \quad x^*(t) \le M_1 + \varepsilon, \\
m_2 - \varepsilon &\le y(t), \quad y^*(t) \le M_2 + \varepsilon, \\
m_3 - \varepsilon &\le z(t), \quad z^*(t) \le M_3 + \varepsilon.
\end{aligned} \tag{2.8}$$

Therefore, for $t \ge T_1$, it follows from (2.8) that

$$\begin{aligned}
D^+V(t) &\le -\Big[\mu_1 b_1(t) - \frac{\mu_1 c_1(t) z(t)}{d_1^2(t)} - \frac{\mu_2 d_4(t) y(t)}{d_3^2(t)} - \frac{\mu_3 c_2(t)}{d_1(t)}\Big]|x^*(t)-x(t)| \\
&\quad -\Big[-\frac{\mu_1 c_1(t) d_2(t) z(t)}{d_1^2(t)} + \frac{\mu_2}{d_3(t)+d_4(t)x^*(t)} - \frac{\mu_3 c_2(t) d_2(t) x(t)}{d_1^2(t)}\Big]|y^*(t)-y(t)| \\
&\quad -\Big[\mu_3 b_2(t) - \frac{\mu_1 c_1(t)}{d_1(t)}\Big]|z^*(t)-z(t)| \\
&\le -\Big[\mu_1 b_1(t) - \frac{\mu_1 c_1(t)(M_3+\varepsilon)}{d_1^2(t)} - \frac{\mu_2 d_4(t)(M_2+\varepsilon)}{d_3^2(t)} - \frac{\mu_3 c_2(t)}{d_1(t)}\Big]|x^*(t)-x(t)| \\
&\quad -\Big[-\frac{\mu_1 c_1(t) d_2(t)(M_3+\varepsilon)}{d_1^2(t)} + \frac{\mu_2}{d_3(t)+d_4(t)(M_1+\varepsilon)} - \frac{\mu_3 c_2(t) d_2(t)(M_1+\varepsilon)}{d_1^2(t)}\Big] \\
&\quad \cdot|y^*(t)-y(t)| - \Big[\mu_3 b_2(t) - \frac{\mu_1 c_1(t)}{d_1(t)}\Big]|z^*(t)-z(t)| \\
&\le -\delta\big[|x^*(t)-x(t)| + |y^*(t)-y(t)| + |z^*(t)-z(t)|\big].
\end{aligned}$$

Integrating the above inequality, we have that

$$V(t)+\delta\int_{T_1}^{t}\big[|x^*(t)-x(t)|+|y^*(t)-y(t)|+|z^*(t)-z(t)|\big]\mathrm{d}s\le V(T_1)<+\infty.$$

Therefore,

$$\limsup_{t\to\infty}\int_{T_1}^{t}\big[|x^*(t)-x(t)|+|y^*(t)-y(t)|+|z^*(t)-z(t)|\big]\mathrm{d}s<\frac{V(T_1)}{\delta}<+\infty.$$

From the above inequality one could easily deduce that

$$\lim_{t\to+\infty}|x^*(t)-x(t)|=0,\quad \lim_{t\to+\infty}|y^*(t)-y(t)|=0,\quad \lim_{t\to+\infty}|z^*(t)-z(t)|=0.$$

This shows that system (1.3) is globally attractive. We complete the proof.

3 Almost Periodic Solution

This section deals with the almost periodic solution of system (1.3). To do so, we further assume that:

(H) $a_1(t),a_2(t),a_3(t),b_1(t),b_2(t),c_1(t),\ c_2(t),d_1(t),d_2(t),d_3(t)$ and $d_4(t)$ are all continuous nonnegative almost periodic functions defined on $[0,+\infty)$, $a_1(t)$, $a_2(t)$, $a_3(t),b_1(t),b_2(t)$ are strictly positive.

Let $x(t)=\mathrm{e}^{\bar{x}(t)}$, $y(t)=\mathrm{e}^{\bar{y}(t)}$, $z(t)=\mathrm{e}^{\bar{z}(t)}$. Then system (1.3) can be revised as

$$\begin{cases}\dot{\bar{x}}(t)=a_1(t)-b_1(t)\mathrm{e}^{\bar{x}(t)}-\dfrac{c_1(t)\mathrm{e}^{\bar{z}(t)}}{d_1(t)+\mathrm{e}^{\bar{x}(t)}+d_2(t)\mathrm{e}^{\bar{y}(t)}},\\[2mm] \dot{\bar{y}}(t)=a_2(t)-\dfrac{\mathrm{e}^{\bar{y}(t)}}{d_3(t)+d_4(t)\mathrm{e}^{\bar{x}(t)}},\\[2mm] \dot{\bar{z}}(t)=-a_3(t)-b_2(t)\mathrm{e}^{\bar{z}(t)}+\dfrac{c_2(t)\mathrm{e}^{\bar{x}(t)}}{d_1(t)+\mathrm{e}^{\bar{x}(t)}+d_2(t)\mathrm{e}^{\bar{y}(t)}}.\end{cases}\tag{3.1}$$

By the relationship of systems (1.3) and (3.1), one could easily obtain following results from Lemmas 2.2 and 2.3.

Lemma 3.1 *If $m_1>0$ and $m_3>0$, then the set S_1 defined by*

$$S_1=\{F=(x,y,z)\in R^3|\ \ln m_1\le\bar{x}(t)\le\ln M_1,\ln m_2\le\bar{y}(t)\le\ln M_2,\ln m_3\le\bar{z}(t)\le\ln M_3\}$$

is invariant with respect to (3.1).

Lemma 3.2 *Assume that $m_1>0$ and $m_3>0$. Let $\overline{W}(t)=(\bar{x}(t),\bar{y}(t),\bar{z}(t))$ be any solution of system* (3.1), *then we have*

$$\ln m_1\le\liminf_{t\to\infty}\bar{x}(t)\le\limsup_{t\to\infty}\bar{x}(t)\le\ln M_1,$$
$$\ln m_2\le\liminf_{t\to\infty}\bar{y}(t)\le\limsup_{t\to\infty}\bar{y}(t)\le\ln M_2,$$
$$\ln m_3\le\liminf_{t\to\infty}\bar{z}(t)\le\limsup_{t\to\infty}\bar{z}(t)\le\ln M_3.$$

It is obvious that the existence of almost periodic solution of system (3.1) is equivalent to that of system (1.3).

Consider the following ordinary differential equation:

$$\dot{X} = f(t, X), \quad f(t, X) \in C(R \times D, R^n), \tag{3.2}$$

where D is an open set in R^n, $f(t, X)$ is almost periodic in t uniformly with respect to $x \in D$.

Lemma 3.3[21] *Suppose that there exists a Lyapunov function $V(t, x, y)$ defined on $[0, +\infty) \times D \times D$, which satisfies the following conditions:*

(1) $\alpha(\|x - y\|) \leq V(t, x, y) \leq \beta(\|x - y\|)$, *where $\alpha(\gamma)$ and $\beta(\gamma)$ are continuous, increasing and positive definite;*

(2) $|V(t, x_1, y_1) - V(t, x_2, y_2)| \leq K\{\| x_1 - x_2\| + \| y_1 - y_2\|\}$, *where $K > 0$ is a constant;*

(3) $\dot{V}(t, x, y) \leq -\mu V(|x - y|)$, *where $\mu > 0$ is a constant.*

Moreover, suppose that system (3.2) has a solution that remains in a compact set $S \subset D$ for all $t \geq t_0 \geq 0$. Then system (3.2) has a unique almost periodic solution in S, which is uniformly asymptotically stable in D.

Theorem 3.1 *In addition to assumption (H), assume further that the conditions of Theorem 2.2 hold. Then system (1.3) admits a unique globally attractive strictly positive almost periodic solution.*

Proof For $(X, Y, Z) \in R^3_+$, we define $\|X, Y, Z\| = |X| + |Y| + |Z|$. We first shows that system (3.1) has a unique almost periodic solution that is uniformly asymptotically stable in S_1. Consider the product system of system (3.1):

$$\begin{cases} \dot{\bar{x}}(t) = a_1(t) - b_1(t)\mathrm{e}^{\bar{x}(t)} - \dfrac{c_1(t)\mathrm{e}^{\bar{z}(t)}}{d_1(t) + \mathrm{e}^{\bar{x}(t)} + d_2(t)\mathrm{e}^{\bar{y}(t)}}, \\ \dot{\bar{y}}(t) = a_2(t) - \dfrac{\mathrm{e}^{\bar{y}(t)}}{d_3(t) + d_4(t)\mathrm{e}^{\bar{x}(t)}}, \\ \dot{\bar{z}}(t) = -a_3(t) - b_2(t)\mathrm{e}^{\bar{z}(t)} + \dfrac{c_2(t)\mathrm{e}^{\bar{x}(t)}}{d_1(t) + \mathrm{e}^{\bar{x}(t)} + d_2(t)\mathrm{e}^{\bar{y}(t)}}; \\ \dot{\bar{x}}^*(t) = a_1(t) - b_1(t)\mathrm{e}^{\bar{x}^*(t)} - \dfrac{c_1(t)\mathrm{e}^{\bar{z}^*(t)}}{d_1(t) + \mathrm{e}^{\bar{x}^*(t)} + d_2(t)\mathrm{e}^{\bar{y}^*(t)}}, \\ \dot{\bar{y}}^*(t) = a_2(t) - \dfrac{\mathrm{e}^{\bar{y}^*(t)}}{d_3(t) + d_4(t)\mathrm{e}^{\bar{x}^*(t)}}, \\ \dot{\bar{z}}^*(t) = -a_3(t) - b_2(t)\mathrm{e}^{\bar{z}^*(t)} + \dfrac{c_2(t)\mathrm{e}^{\bar{x}^*(t)}}{d_1(t) + \mathrm{e}^{\bar{x}^*(t)} + d_2(t)\mathrm{e}^{\bar{y}^*(t)}}. \end{cases} \tag{3.3}$$

Suppose that $\overline{W}(t) = (\bar{x}(t), \bar{y}(t), \bar{z}(t))$, $\overline{Q}(t) = (\bar{x}^*(t), \bar{y}^*(t), \bar{z}^*(t))$ are any two solutions of system (3.1) defined on $[0, +\infty) \times S_1 \times S_1 \times S_1$. Consider a Lyapunov function defined on $[0,+\infty)\times S_1\times S_1\times S_1$ as follows:

$$V(t,\overline{W}(t),\overline{Q}(t))=\mu_1|\bar{x}_1(t)-\bar{x}_1^*(t)|+\mu_2|\bar{y}_1(t)-\bar{y}_2^*(t)|+\mu_3|\bar{z}_3(t)-\bar{z}_3^*(t)|.$$

It then follows

$$A\|\overline{W}(t)-\overline{Q}(t)\|\le V(t,\overline{W}(t),\overline{Q}(t))\le B\|\overline{W}(t)-\overline{Q}(t)\|,$$

where $A=\min\{\mu_1,\mu_2,\mu_3\}$, $B=\max\{\mu_1,\mu_2,\mu_3\}$, thus condition (1) of Lemma 3.3 is satisfied.

In addition

$$\begin{aligned}
&|V(t,\overline{W}_1(t),\overline{Q}_1(t))-V(t,\overline{W}_2(t),\overline{Q}_2(t))|\\
=&\big|(\mu_1|\bar{x}_1(t)-\bar{x}_1^*(t)|+\mu_2|\bar{y}_1(t)-\bar{y}_1^*(t)|+\mu_3|\bar{z}_1(t)-\bar{z}_1^*(t)|)\\
&-(\mu_1|\bar{x}_2(t)-\bar{x}_2^*(t)|+\mu_2|\bar{y}_2(t)-\bar{y}_2^*(t)|+\mu_3|\bar{z}_2(t)-\bar{z}_2^*(t)|)\\
&-(\mu_1|\bar{x}_3(t)-\bar{x}_3^*(t)|+\mu_2|\bar{y}_3(t)-\bar{y}_3^*(t)|+\mu_3|\bar{z}_3(t)-\bar{z}_3^*(t)|)\big|\\
\le&\mu_1|\bar{x}_1(t)-\bar{x}_1^*(t)|+\mu_1|\bar{x}_2(t)-\bar{x}_2^*(t)|+\mu_1|\bar{x}_3(t)-\bar{x}_3^*(t)|\\
&+\mu_2|\bar{y}_1(t)-\bar{y}_1^*(t)|+\mu_2|\bar{y}_2(t)-\bar{y}_2^*(t)|+\mu_2|\bar{y}_3(t)-\bar{y}_3^*(t)|\\
&+\mu_3|\bar{z}_1(t)-\bar{z}_1^*(t)|+\mu_3|\bar{z}_2(t)-\bar{z}_2^*(t)|+\mu_3|\bar{z}_3(t)-\bar{z}_3^*(t)|\\
\le&B\{\|\overline{W}_1(t)-\overline{W}_2(t)\|+\|\overline{Q}_1(t)-\overline{Q}_2(t)\|\},
\end{aligned}$$

where $B=\max\{\mu_1,\mu_2,\mu_3\}$, thus condition (2) of Lemma 3.3 is also satisfied.

Finally, calculating the right derivative $D^+V(t)$ of $V(t)$ along the solutions of system (3.3), using Lemma 3.2, similar to the analysis of Theorem 2.1, we can obtain:

$$\begin{aligned}
D^+V(t)\le&-\Big[\mu_1b_1(t)-\frac{\mu_2c_1(t)M_3}{d_1^2(t)}-\frac{\mu_2d_4(t)M_2}{d_3^2(t)}-\frac{\mu_3c_2(t)}{d_1(t)}\Big]|\mathrm{e}^{\bar{x}^*(t)}-\mathrm{e}^{\bar{x}(t)}|\\
&-\Big[-\frac{\mu_1c_1(t)d_2(t)M_3}{d_1^2(t)}+\frac{\mu_2}{d_3(t)+d_4(t)M_1}-\frac{\mu_3c_2(t)d_2(t)M_1}{d_1^2(t)}\Big]|\mathrm{e}^{\bar{y}^*(t)}-\mathrm{e}^{\bar{y}(t)}|\\
&-\Big[\mu_3b_2(t)-\frac{\mu_1c_1(t)}{d_1(t)}\Big]|\mathrm{e}^{\bar{z}^*(t)}-\mathrm{e}^{\bar{z}(t)}|\\
\le&-\delta\big[|\mathrm{e}^{\bar{x}^*(t)}-\mathrm{e}^{\bar{x}(t)}|+|\mathrm{e}^{\bar{y}^*(t)}-\mathrm{e}^{\bar{y}(t)}|+|\mathrm{e}^{\bar{z}^*(t)}-\mathrm{e}^{\bar{z}(t)}|\big].
\end{aligned}$$

Note that

$$\begin{aligned}
\mathrm{e}^{\bar{x}^*(t)}-\mathrm{e}^{\bar{x}(t)}&=\mathrm{e}^{\bar{\zeta}_1}(\bar{x}^*(t)-\bar{x}(t)),\\
\mathrm{e}^{\bar{y}^*(t)}-\mathrm{e}^{\bar{y}(t)}&=\mathrm{e}^{\bar{\zeta}_2}(\bar{y}^*(t)-\bar{y}(t)),\\
\mathrm{e}^{\bar{z}^*(t)}-\mathrm{e}^{\bar{z}(t)}&=\mathrm{e}^{\bar{\zeta}_3}(\bar{z}^*(t)-\bar{z}(t)).
\end{aligned}$$

Here, $\bar{\zeta}_1$ is a bounded function between $\bar{x}^*(t)$ and $\bar{x}(t)$, $\bar{\zeta}_2$ is a bounded function between $\bar{y}^*(t)$ and $\bar{y}(t)$ and $\bar{\zeta}_3$ is a bounded function between $\bar{z}^*(t)$ and $\bar{z}(t)$. Then we have

$$
\begin{aligned}
D^+V(t) &\le -\delta[m_1|x^*(t)-x(t)|+m_2|y^*(t)-y(t)|+m_3|z^*(t)-z(t)|] \\
&\le -m\delta\{|x^*(t)-x(t)|+|y^*(t)-y(t)|+|z^*(t)-z(t)|\} \\
&\le \frac{-m\delta}{B}V(t,\overline{W}_1,\overline{W}_2),
\end{aligned}
$$

where $m=\min\{m_1,m_2,m_3\}$, $B=\max\{\mu_1,\mu_2,\mu_3\}$. Hence, condition (3) of Lemma 3.3 is also satisfied.

The above analysis shows that all the conditions of Lemma 3.3 hold. Thus, system (3.1) has a unique almost periodic solution $(\tilde{x}^*,\tilde{y}^*,\tilde{z}^*)$ which is uniformly asymptotically stable in S_1. Hence, system (1.3) has a unique positive almost periodic solution $(\mathrm{e}^{\tilde{x}^*},\mathrm{e}^{\tilde{y}^*},\mathrm{e}^{\tilde{z}^*})$, which is uniformly asymptotically stable in S. The proof is complete.

4 Example

In this section, we shall give an example to illustrate the feasibility of the main result.

Example 4.1 Considering the following predator-prey-mutualist system:

$$
\begin{cases}
\dot{x}=x\Big(\big(0.9+0.1\cos(3\sqrt{3}t)\big)-1.5x-\dfrac{0.5z}{3+x+1.4y}\Big); \\
\dot{y}=y\Big(\big(0.9+0.1\sin(5\sqrt{5}t)\big)-\dfrac{y}{0.2+0.5x}\Big); \\
\dot{z}=z\Big(-\big(0.03+0.01\cos(7\sqrt{3}t)\big)-\big(0.19+0.1\cos(6\sqrt{5}t)\big)z+\dfrac{0.4x}{3+x+1.4y}\Big).
\end{cases}
\tag{4.1}
$$

Comparing with system (1.3), we have $a_1(t)=0.9+0.1\cos(\sqrt{3}t)$, $b_1(t)=1.5$, $c_1(t)=0.5$, $c_2(t)=0.4$, $a_2(t)=0.9+0.1\sin(\sqrt{5}t)$, $a_3(t)=0.03+0.01\cos(\sqrt{7}t)$, $b_2(t)=0.19+0.1\cos(\sqrt{11}t)$, $d_1(t)=3$, $d_2(t)=1.4$, $d_3(t)=0.2$, $d_4(t)=0.5$.

Let $(x(t),y(t),z(t))$ be any positive solution of system (4.1), then by simple computation, we have

$$
\begin{aligned}
0.4652 \le \liminf_{t\to\infty} x(t) \le \limsup_{t\to\infty} x(t) \le 0.6667, \\
0.16 \le \liminf_{t\to\infty} y(t) \le \limsup_{t\to\infty} y(t) \le 0.5334, \\
0.0075 \le \liminf_{t\to\infty} z(t) \le \limsup_{t\to\infty} z(t) \le 0.7655.
\end{aligned}
$$

From the above inequality, we also note that condition (I) of Theorem 2.2 holds. Letting $\mu_1=8$, $\mu_2=1$, $\mu_3=15$, $\delta=\frac{1}{100}$, one could easily verify that

$$
\varphi(t)=\mu_1b_1(t)-\frac{\mu_1c_1(t)M_3}{d_1^2(t)}-\frac{\mu_2d_4(t)M_2}{d_3^2(t)}-\frac{\mu_3c_2(t)}{d_1(t)}=2.354>\frac{1}{100};
$$

$$\psi(t) = \frac{\mu_2}{d_3(t) + d_4(t)M_1} - \frac{\mu_1 c_1(t) d_2(t) M_3}{d_1^2(t)} - \frac{\mu_3 c_2(t) d_2(t) M_1}{d_1^2(t)} = 0.7779 > \frac{1}{100};$$
$$\phi(t) = \mu_3 b_2(t) - \frac{\mu_1 c_1(t)}{d_1(t)} = 0.0164 > \frac{1}{100}.$$

The above three inequalities show that condition (II) of Theorem 2.2 holds. Thus, from Theorem 3.1, system (4.1) admits a unique globally attractive positive almost periodic solution $(x^*(t), y^*(t), z^*(t))$.

Numerical simulation (Figure 1) strongly supports our main result.

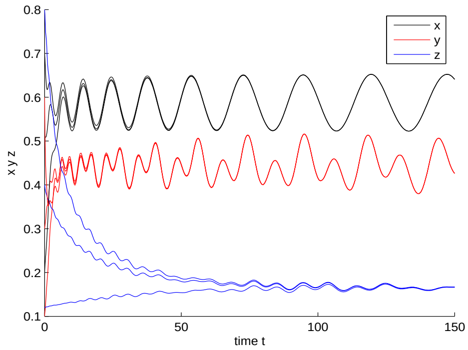

Figure 1: Dynamic behavior of solutions $(x(t), y(t), z(t))$ of system (4.1) with the initial conditions $\big(x(0), y(0), z(0)\big) = (0.2, 0.6, 0.8), (0.5, 0.3, 0.4)$ and $(0.7, 0.1, 0.12)$, respectively.

5 Conclusion

In this paper, we study an almost periodic predator-prey-mutualist system. Some sufficient conditions which guarantee the existence of the unique globally attractive positive almost periodic solution of system (1.3). Example shows the feasibility of our main result. Our result indicates that if the death rate of the predator specie z is small enough, the density restriction of z is large enough and the cooperate effect between species x and y is very strong, then there exists a unique globally attractive positive almost periodic solution of system (1.3).

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