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WEAK AND STRONG CONVERGENCE THEOREMS FOR SPLIT GENERALIZED MIXED EQUILIBRIUM PROBLEM*†

Jingni Ye‡

(*Zhicheng College, Fuzhou University, Fujian 350002, PR China*)

Abstract

The purpose of this paper is to introduce a split generalized mixed equilibrium problem (SGMEP) and consider some iterative sequences to find a solution of the generalized mixed equilibrium problem such that its image under a given bounded linear operator is a solution of another generalized mixed equilibrium problem. We obtain some weak and strong convergence theorems.

Keywords split generalized mixed equilibrium problem; weak convergence; strong convergence; fixed point

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1 Introduction and Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and C be a nonempty closed convex subset of H. Let f be a bi-function from $C\times C$ to R and $\varphi : C\to R$ be a function, where R is the set of real numbers. Let $B : C\to H$ be a nonlinear mapping. Then we consider the following generalized mixed equilibrium problem: There exists an $x\in C$, such that

$$f(x,y)+\varphi(y)-\varphi(x)+\langle Bx,y-x\rangle\geq 0,\quad \text{for any } y\in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $GMEP(f,\varphi,B)$.

If $B=0$, problem (1.1) becomes the following mixed equilibrium problem: There exists an $x\in C$, such that

$$f(x,y)+\varphi(y)-\varphi(x)\geq 0,\quad \text{for any } y\in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $MEP(f,\varphi)$.

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‡Corresponding author. E-mail: 46793299@qq.com

If $\varphi = 0$, problem (1.1) reduces to the following generalized equilibrium problem: There exists an $x \in C$, such that

$$f(x, y) + \langle Bx, y - x\rangle \geq 0, \quad \text{for any } y \in C. \tag{1.3}$$

The set of solutions of (1.3) is denoted by $GEP(f, B)$.

If $\varphi = 0$ and $B = 0$, problem (1.1) becomes the following equilibrium problem: There exists an $x \in C$, such that

$$f(x, y) \geq 0, \quad \text{for any } y \in C. \tag{1.4}$$

The set of solutions of (1.4) is denoted by $EP(f)$.

Equilibrium problem is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, mini or max problems, Nash equilibrium problem in noncooperative games and others; see for instance [1-20].

In 2012, Zhenhua He [12] proposed a new equilibrium problem which is called split equilibrium problem (SEP). Let E_1 and E_2 be two real Banach spaces, C be a closed convex subset of E_1, K be a closed convex subset of E_2, $A : E_1 \to E_2$ be a bounded linear operator, f be a bi-function from $C \times C$ into R and g be a bi-function from $K \times K$ into R. The SEP is to find an element $x^* \in C$, such that

$$f(x^*, y) \geq 0, \quad \text{for any } y \in C,$$

and such that $u := Ax^* \in K$ satisfying

$$g(u, v) \geq 0, \quad \text{for any } v \in K.$$

Inspired and motivated by the above works, we propose a split generalized mixed equilibrium problem (SGMEP). Let E_1 and E_2 be two real Banach spaces, E_1^* and E_2^* denote the dual of E_1 and E_2, respectively, C be a closed convex subset of E_1, K be a closed convex subset of E_2, $A : E_1 \to E_2$ be a bounded linear operator, f be a bi-function from $C \times C$ into R, g be a bi-function from $K \times K$ into R, $B : C \to E_1^*$ and $S : K \to E_2^*$ be two mappings, $\varphi : C \to R$ and $\psi : K \to R$ be two functions. The SGMEP is to find an element $p \in C$ such that

$$f(p, y) + \varphi(y) - \varphi(p) + \langle Bp, y - p\rangle \geq 0, \quad \text{for any } y \in C, \tag{1.5}$$

and that $u := Ap \in K$ satisfies

$$g(u, v) + \psi(v) - \psi(u) + \langle Su, v - u\rangle \geq 0, \quad \text{for any } v \in K. \tag{1.6}$$

For convenience, we denote the solution set of the SGMEP by Ω, that is, $\Omega = \{x \in GMEP(f, \varphi, B) : Ax \in GMEP(g, \psi, S)\}$.

Now, we give two examples to show $\Omega \neq \emptyset$.

Example 1.1 Let $E_1 = E_2 = R$, $C := [1, +\infty)$, $K := [-2, +\infty)$. Let $Ax = -2x$ for all $x \in R$, then A is a bounded linear operator. Let $f(x, y) = y - x$, $\varphi(x) = x$, $Bx = x^2$, $g(u, v) = v - u$, $\psi(u) = -2u$, $Su = u^2$. Clearly, $GMEP(f, \varphi, B) = \{1\}$ and $A(1) = -2 \in GMEP(g, \psi, S)$. So $\Omega = \{x \in GMEP(f, \varphi, B) : Ax \in GMEP(g, \psi, S)\} \neq \emptyset$.

Example 1.2 Let $E_1 = R^2$ with the norm $\|\alpha\| = (a_1^2+a_2^2)^{\frac{1}{2}}$ for each $\alpha = (a_1, a_2)$ and $E_2 = R$ with the standard norm $|\cdot|$. Let $C := \{\alpha = (a_1, a_2) \in R^2 |\, a_2 - a_1 \geq 1\}$ and $K := [1, +\infty)$. Define $f(p, y) = -[p_1^2(y_1 - p_1) + p_2^2(y_2 - p_2)]$, $\varphi(p) = p_2 - p_1$, $Bp = (p_1^2, p_2^2)$, where $p = (p_1, p_2)$, $y = (y_1, y_2) \in C$. For each $\alpha = (a_1, a_2) \in E_1$, let $A\alpha = a_2 - a_1$, then A is a bounded linear operator from E_1 into E_2. Next we define $g(u, v) = v - u$, $Su = u^2$, $\psi(v) = -v$ for all $u, v \in K$. Direct computation shows that $GMEP(f, \varphi, B) = \{p = (p_1, p_2) |\, p_2 - p_1 = 1\}$ and $Ap = p_2 - p_1 = 1 \in GMEP(g, \psi, S)$. So $\Omega = \{x \in GMEP(f, \varphi, B) : Ax \in GMEP(g, \psi, S)\} \neq \emptyset$.

Remark If $B = S = \theta$, the SGMEP reduces to the split mixed equilibrium problem (SMEP); if $\varphi = \psi = 0$, the SGMEP becomes the split generalized equilibrium problem (SGEP); if $B = S = \theta$ and $\varphi = \psi = 0$, the SGMEP reduces to the split equilibrium problem (SEP) (see [12]).

In this paper, we construct two iterative algorithms to solve the SGMEP. Some weak and strong convergence theorems are established. The results obtained in this paper improve and extend the corresponding results announced by many others.

2 Preliminaries

In this paper, we denote the sets of positive integers and real numbers by N and R, respectively. We also denote by "$\to$" and "$\rightharpoonup$" the strong convergence and weak convergence, respectively.

Recall that the mapping $S : C \to C$ is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \text{for any } x, y \in C.$$

We denote by $Fix(S)$ the sets of fixed points of the mapping S.

A mapping $B : C \to H$ is said to be α-inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Bx - By, x - y\rangle \geq \alpha\|Bx - By\|^2, \quad \text{for any } x, y \in C.$$

For all $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that $\|x - P_C(x)\| \leq \|x - y\|$ for all $y \in C$. The mapping P_C is called the metric projection of H onto C. It is also known that P_C satisfying

$$\langle x - P_C(x), P_C(x) - y\rangle \geq 0,$$

for all $x \in H$ and $y \in C$.

Lemma 2.1 *Let C be a closed convex subset of H. Define a mapping P_C as the metric projection from H onto C. Then P_C has the following characters:*

(1) $\langle x-y, P_C(x)-P_C(y)\rangle \geq \|P_C(x)-P_C(y)\|^2$, *for any* $x,y \in H$;

(2) *for* $x\in H$ *and* $z\in C$, $z=P_C(x)$ *if and only if* $\langle x-z, z-y\rangle \geq 0$, *for any* $y\in C$;

(3) *for* $x \in H$ *and* $y \in C$, $\|y-P_C(x)\|^2+\|x-P_C(x)\|^2 \leq \|x-y\|^2$.

Definition 2.1 A Banach space $(X, \|\cdot\|)$ is said to satisfy Opial's condition if, for each sequence $\{x_n\}$ in X which converges weakly to a point $x \in X$, we have

$$\liminf_{n\to\infty}\|x_n-x\| < \liminf_{n\to\infty}\|x_n-y\|, \quad \text{for any } y\in X,\ y\neq x.$$

It is well known that each Hilbert space satisfies Opial's condition.

Lemma 2.2[3] *Let H be a real Hilbert space, C be nonempty closed convex subset of H and $S: C\to C$ be a nonexpansive mapping. Then the mapping $I-S$ is demiclosed at zero, that is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $x_n - Sx_n \to 0$, then $x \in Fix(S)$.*

Lemma 2.3 *Let H be a real Hilbert space. Then for any $x,y\in H$, we have*

(1) $\|\alpha x+(1-\alpha)y\|^2=\alpha\|x\|^2+(1-\alpha)\|y\|^2-\alpha(1-\alpha)\|x-y\|^2,\ \alpha\in(0,1)$;

(2) $\langle x,y\rangle=\frac{1}{2}(\|x\|^2+\|y\|^2-\|x-y\|^2)$.

Let H_1 and H_2 be two Hilbert spaces. The operator A from H_1 into H_2 and the operator A^* from H_2 into H_1 are two bounded linear operators. A^* is called the adjoint operator of A, if for all $x\in H_1$, $y\in H_2$, A^* satisfies $\langle Ax,y\rangle=\langle x,A^*y\rangle$. Then A^* has the following characters:

(1) $\|A^*\|=\|A\|$;

(2) A^* is a unique adjoint operator of A.

For solving the split generalized mixed equilibrium problem, we assume that the function $f: C\times C\to R$ satisfies the following conditions:

(A1) $f(x,x)=0$, for all $x\in C$;

(A2) f is monotone, that is $f(x,y)+f(y,x)\leq 0$, for all $x,y\in C$;

(A3) for each $y\in C,\ x\mapsto f(x,y)$, is weakly upper semicontinuous;

(A4) for each $x\in C,\ y\mapsto f(x,y)$ is convex and lower semicontinuous;

(B1) for each $x\in H$ and $r>0$, there exists a bounded subset $D_x\subseteq C$ and $y_x\in C$ such that for any $z\in C\backslash D_x$,

$$f(z,y_x)+\varphi(y_x)-\varphi(z)+\frac{1}{r}\langle y_x-z, z-x\rangle<0;$$

(B2) C is a bounded set.

Lemma 2.4[2,14] *Let C be a nonempty closed convex subset of H, f be a bi-function from $C\times C$ to R satisfying* (A1)-(A4) *and $\varphi: C\to R$ be a proper lower semicontinuous and convex function. For $r>0$ and $x\in H$, define a mapping $T_r^{f,\varphi}: H\to C$ as follows:*

$$T_r^{f,\varphi}(x)=\{z\in C: f(z,y)+\varphi(y)-\varphi(z)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0,\ for\ any\ y\in C\},$$

for all $x\in H$. *Assume that either* (B1) *or* (B2) *holds. Then the following results hold:*

(1) *For each* $x\in H, T_r^{f,\varphi}\neq\emptyset$;

(2) $T_r^{f,\varphi}$ *is single-valued;*

(3) $T_r^{f,\varphi}$ *is firmly non-expansive, that is for any* $x,y\in H$

$$\|T_r^{f,\varphi}(x)-T_r^{f,\varphi}(y)\|^2\leq\langle T_r^{f,\varphi}(x)-T_r^{f,\varphi}(y),x-y\rangle;$$

(4) $Fix(T_r^{f,\varphi})=MEP(f,\varphi)$;

(5) $MEP(f,\varphi)$ *is closed and convex.*

3 Main Results

Theorem 3.1(Weak convergence theorem) *Let* C *be a nonempty closed convex subset of* H_1, K *be a nonempty closed convex subset of* H_2, *where* H_1 *and* H_2 *are two real Hilbert spaces,* $f:C\times C\to R$ *and* $g:K\times K\to R$ *be two bi-functions which satisfy* (A1)-(A4), $\varphi:C\to R$ *be a lower semicontinuous and convex function,* $\psi:K\to R$ *be a lower semicontinuous and convex function,* $A:H_1\to H_2$ *be a bounded linear operator with the adjoint operator* A^*, $B:C\to H_1$ *be an* α*-inverse-strongly monotone mapping and* $S:K\to H_2$ *be a* β*-inverse-strongly monotone mapping. Assume that* $GMEP(f,\varphi,B)\neq\emptyset$ *and* $GMEP(g,\psi,S)\neq\emptyset$. *Let* $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated in the following manner:*

$$\begin{cases}x_1\in C,\\ f(u_n,y)+\varphi(y)-\varphi(u_n)+\frac{1}{r}\langle y-u_n,u_n-x_n\rangle+\langle Bx_n,y-u_n\rangle\geq 0,\\ g(w_n,z)+\psi(z)-\psi(w_n)+\frac{1}{r}\langle z-w_n,w_n-Au_n\rangle+\langle S(Au_n),z-w_n\rangle\geq 0,\\ x_{n+1}=\alpha_nx_n+(1-\alpha_n)P_C(u_n+\mu A^*(w_n-Au_n)),\end{cases}\tag{3.1}$$

for any $y\in C$, $z\in K$, $n\in N$, *where* $r\in(0,a)$, $a=\min\{2\alpha,2\beta\}$, $\alpha_n\in(0,1)$ *and* $\mu\in(0,\frac{1}{\|A^*\|^2})$ *are constants. Suppose that* $\Omega=\{x\in GMEP(f,\varphi,B):Ax\in GMEP(g,\psi,S)\}\neq\emptyset$. *For* f,φ *and* C, *assume that either* (B1) *or* (B2) *holds. For* g,ψ *and* K, *assume that either* (B1) *or* (B2) *also holds, then the sequences* $\{x_n\}$ *and* $\{u_n\}$ *converge weakly to an element* $p\in GMEP(f,\varphi,B)$, *while* $\{w_n\}$ *converges weakly to* $Ap\in GMEP(g,\psi,S)$.

Proof Let $x^*\in\Omega$, namely $x^*\in GMEP(f,\varphi,B)$ and $Ax^*\in GMEP(g,\psi,S)$. By Lemma 2.4, it follows that

$$\begin{aligned}T_r^{f,\varphi}(I-rB)(x)=\{z\in C: &f(z,y)+\varphi(y)-\varphi(z)\\ &+\frac{1}{r}\langle y-z,z-(I-rB)x\rangle\geq 0,\text{ for any } y\in C\},\end{aligned}$$

namely,

$$T_r^{f,\varphi}(I-rB)(x)=\{z\in C: f(z,y)+\varphi(y)-\varphi(z) \\ +\frac{1}{r}\langle y-z,z-x\rangle+\langle Bx,y-z\rangle\geq 0,\text{ for any } y\in C\}.$$

Hence, we obtain that $GMEP(f,\varphi,B)=Fix(T_r^{f,\varphi}(I-rB))$. We also have that $GMEP(g,\psi,S)=Fix(T_r^{g,\psi}(I-rS))$. From (3.1), we get

$$u_n=T_r^{f,\varphi}(I-rB)(x_n), \tag{3.2}$$

$$w_n=T_r^{g,\psi}(I-rS)(Au_n), \tag{3.3}$$

$$x^*=T_r^{f,\varphi}(I-rB)x^*,\quad Ax^*=T_r^{g,\psi}(I-rS)Ax^*. \tag{3.4}$$

For any $x,y\in C$, we see that

$$\begin{aligned}\|(I-rB)x-(I-rB)y\|^2&=\|(x-y)-r(Bx-By)\|^2\\&=\|x-y\|^2-2r\langle x-y,Bx-By\rangle+r^2\|Bx-By\|^2\\&\leq\|x-y\|^2-2r\alpha\|Bx-By\|^2+r^2\|Bx-By\|^2\\&=\|x-y\|^2-r(2\alpha-r)\|Bx-By\|^2\\&\leq\|x-y\|^2.\end{aligned}\tag{3.5}$$

So, $I-rB$ is nonexpansive. In a similar way, we can deduce that $I-rS$ is nonexpansive. By (3.2),(3.3) and (3.3), we notice that

$$\|u_n-x^*\|=\|T_r^{f,\varphi}(I-rB)x_n-x^*\|\leq\|x_n-x^*\|, \tag{3.6}$$

$$\|w_n-Ax^*\|=\|T_r^{g,\psi}(I-rS)Au_n-Ax^*\|\leq\|Au_n-Ax^*\|. \tag{3.7}$$

From (3.5), we have

$$\begin{aligned}\|u_n-x^*\|^2&=\|T_r^{f,\varphi}(I-rB)x_n-x^*\|^2\\&\leq\|(I-rB)x_n-(I-rB)x^*\|^2\\&\leq\|x_n-x^*\|^2-r(2\alpha-r)\|Bx_n-Bx^*\|^2,\end{aligned}\tag{3.8}$$

$$\begin{aligned}\|w_n-Ax^*\|^2&=\|T_r^{g,\psi}(I-rS)Au_n-Ax^*\|^2\\&\leq\|(I-rS)Au_n-(I-rS)Ax^*\|^2\\&\leq\|Au_n-Ax^*\|^2-r(2\beta-r)\|S(Au_n)-S(Ax^*)\|^2.\end{aligned}\tag{3.9}$$

By (3.7), we obtain that

$$
\begin{aligned}
&2\mu\langle u_n - x^*, A^*(w_n - Au_n)\rangle \\
&= 2\mu\langle A(u_n - x^*) + (w_n - Au_n) - (w_n - Au_n), w_n - Au_n\rangle \\
&= 2\mu(\langle w_n - Ax^*, w_n - Au_n\rangle - \|w_n - Au_n\|^2) \\
&= 2\mu\Big(\frac{1}{2}\|w_n - Ax^*\|^2 + \frac{1}{2}\|w_n - Au_n\|^2 - \frac{1}{2}\|Au_n - Ax^*\|^2 - \|w_n - Au_n\|^2\Big) \\
&\le 2\mu\Big(\frac{1}{2}\|Au_n - Ax^*\|^2 + \frac{1}{2}\|w_n - Au_n\|^2 - \frac{1}{2}\|Au_n - Ax^*\|^2 - \|w_n - Au_n\|^2\Big) \\
&= 2\mu\Big(\frac{1}{2}\|w_n - Au_n\|^2 - \|w_n - Au_n\|^2\Big) \\
&= -\mu\|w_n - Au_n\|^2. \qquad (3.10)
\end{aligned}
$$

We also have

$$\|A^*(w_n - Au_n)\|^2 \le \|A^*\|^2\|w_n - Au_n\|^2. \qquad (3.11)$$

From (3.1),(3.6),(3,10) and (3.11), we see that

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \|\alpha_n x_n + (1-\alpha_n)P_C(u_n + \mu A^*(w_n - Au_n)) - x^*\|^2 \\
&\le \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|P_C(u_n + \mu A^*(w_n - Au_n)) - P_C x^*\|^2 \\
&\le \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)(\|u_n - x^*\|^2 + \|\mu A^*(w_n - Au_n)\|^2 \\
&\quad +2\mu\langle u_n - x^*, A^*(w_n - Au_n)\rangle) \\
&\le \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)(\|u_n - x^*\|^2 + \mu^2\|A^*\|^2\|w_n - Au_n\|^2 - \mu\|w_n - Au_n\|^2) \\
&= \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|u_n - x^*\|^2 - \mu(1-\alpha_n)(1-\mu\|A^*\|^2)\|w_n - Au_n\|^2 \\
&\le \|x_n - x^*\|^2 - \mu(1-\alpha_n)(1-\mu\|A^*\|^2)\|w_n - Au_n\|^2. \qquad (3.12)
\end{aligned}
$$

Notice that $\mu \in (0, \frac{1}{\|A^*\|^2})$, $\alpha_n \in (0,1)$. It follows from (3.12) that $\lim_{n\to\infty}\|x_n - x^*\|$ exists. So $\{x_n\}$ is bounded and from (3.1), $\{u_n\}$ is also bounded.

Again by (3.12), it implies that

$$\mu(1-\alpha_n)(1-\mu\|A^*\|^2)\|w_n - Au_n\|^2 \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2,$$

hence

$$\lim_{n\to\infty}\|w_n - Au_n\| = 0. \qquad (3.13)$$

Put $z_n = P_C(u_n + \mu A^*(w_n - Au_n))$, for each $n \ge 1$. It follows from (3.6) and (3.12) that

$$\|z_n - x^*\| \le \|u_n - x^*\| \le \|x_n - x^*\|. \qquad (3.14)$$

From (3.12), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \alpha_n\|x_n-x^*\|^2+(1-\alpha_n)\|z_n-x^*\|^2\\ &=\|x_n-x^*\|^2+(1-\alpha_n)(\|z_n-x^*\|^2-\|x_n-x^*\|^2).\end{aligned}$$

This shows that

$$0\le(1-\alpha_n)(\|x_n-x^*\|^2-\|z_n-x^*\|^2)\le\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2.$$

It follows from (3.14) that

$$\lim_{n\to\infty}\|u_n-x^*\|=\lim_{n\to\infty}\|z_n-x^*\|=\lim_{n\to\infty}\|x_n-x^*\|. \tag{3.15}$$

From (3.8), we obtain that

$$\lim_{n\to\infty}\|Bx_n-Bx^*\|=0. \tag{3.16}$$

Applying (3) of Lemma 2.4 and (2) of Lemma 2.3, we have

$$\begin{aligned}\|u_n-x^*\|^2 &= \|T_r^{f,\varphi}(I-rB)x_n-T_r^{f,\varphi}(I-rB)x^*\|^2\\ &\le\langle(I-rB)x_n-(I-rB)x^*,u_n-x^*\rangle\\ &=\frac{1}{2}(\|(I-rB)x_n-(I-rB)x^*\|^2+\|u_n-x^*\|^2\\ &\quad-\|(I-rB)x_n-(I-rB)x^*-(u_n-x^*)\|^2)\\ &\le\frac{1}{2}(\|x_n-x^*\|^2+\|u_n-x^*\|^2-\|x_n-u_n-r(Bx_n-Bx^*)\|^2)\\ &=\frac{1}{2}(\|x_n-x^*\|^2+\|u_n-x^*\|^2-(\|x_n-u_n\|^2\\ &\quad+r^2\|Bx_n-Bx^*\|^2-2r\langle x_n-u_n,Bx_n-Bx^*\rangle)),\end{aligned} \tag{3.17}$$

which yields that

$$\|u_n-x^*\|^2\le\|x_n-x^*\|^2-\|x_n-u_n\|^2+2r\|x_n-u_n\|\|Bx_n-Bx^*\|,$$

namely

$$\|x_n-u_n\|^2\le\|x_n-x^*\|^2-\|u_n-x^*\|^2+2r\|x_n-u_n\|\|Bx_n-Bx^*\|.$$

Further, combining (3.15) with (3.16), we have

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{3.18}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ which converges weakly to $p\in C$. Then $u_{n_j}\rightharpoonup p$ and $Au_{n_j}\rightharpoonup Ap$ by (3.18).

Next we prove $p\in\Omega$. By (3) of Lemma 2.4, we have $GMEP(f,\varphi,B)=Fix(T_r^{f,\varphi}(I-rB))$, $GMEP(g,\psi,S)=Fix(T_r^{g,\psi}(I-rS))$. Since

$$\lim_{j\to\infty}\|x_{n_j}-u_{n_j}\|=\lim_{j\to\infty}\|x_{n_j}-T_r^{f,\varphi}(I-rB)x_{n_j}\|=0,$$

and $T_r^{f,\varphi}(I-rB):C\to C$ is nonexpansive, we have $T_r^{f,\varphi}(I-rB)p=p$ by Lemma 2.2. This shows $p\in GMEP(f,\varphi,B)$. We also can prove $Ap\in GMEP(g,\psi,S)$, similarly.

Finally, we prove $\{x_n\}$ and $\{u_n\}$ converge weakly to $p\in\Omega$, respectively, while $\{w_n\}$ converges weakly to Ap. Assume that there exists another subsequence $\{x_{n_k}\}$ of $\{x_n\}$, such that $\{x_{n_k}\}$ converges weakly to $q\in\Omega$, where $p\neq q$. In view of the Opial's condition, we see that

$$\begin{aligned}\lim_{n\to\infty}\|x_n-q\|&=\liminf_{k\to\infty}\|x_{n_k}-q\|<\liminf_{k\to\infty}\|x_{n_k}-p\|=\lim_{n\to\infty}\|x_n-p\|\\&=\liminf_{j\to\infty}\|x_{n_j}-p\|<\liminf_{j\to\infty}\|x_{n_j}-q\|=\lim_{n\to\infty}\|x_n-q\|.\end{aligned}$$

This is a contradiction, so we have $p=q$. Hence $\{x_n\}$ and $\{u_n\}$ converge weakly to $p\in\Omega$. Furthermore, from (3.13) we notice that $\lim\limits_{n\to\infty}\|w_n-Au_n\|^2=0$, so we get that $Au_n\rightharpoonup Ap$ and $w_n\rightharpoonup Ap$. The proof is completed.

If $B=S=\theta$ in Theorem 3.1, then the split generalized mixed equilibrium problem (SGMEP) is reduced to a split generalized equilibrium problem (SMEP).

Corollary 3.1 *Let C be a nonempty closed convex subset of H_1 and K be a nonempty closed convex subset of H_2, where H_1 and H_2 are two real Hilbert spaces. Let $f:C\times C\to R$ and $g:K\times K\to R$ be two bi-functions which satisfy* (A1)-(A4), *$\varphi:C\to R$ be a lower semicontinuous and convex function, $\psi:K\to R$ be a lower semicontinuous and convex function, $A:H_1\to H_2$ be a bounded linear operator with the adjoint operator A^*. Assume that $MEP(f,\varphi)\neq\emptyset$ and $MEP(g,\psi)\neq\emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated in the following manner:*

$$\begin{cases}x_1\in C,\\ f(u_n,y)+\varphi(y)-\varphi(u_n)+\frac{1}{r}\langle y-u_n,u_n-x_n\rangle\geq 0,\quad y\in C,\\ g(w_n,z)+\psi(z)-\psi(w_n)+\frac{1}{r}\langle z-w_n,w_n-Au_n\rangle\geq 0,\quad z\in K,\\ x_{n+1}=\alpha_nx_n+(1-\alpha_n)P_C(u_n+\mu A^*(w_n-Au_n)),\quad \text{for any } n\in N,\end{cases}$$

where $r>0$ and $\alpha_n\in(0,1)$, $\mu\in(0,\frac{1}{\|A^*\|^2})$ are constants. Suppose that $\Omega=\{x\in MEP(f,\varphi):Ax\in MEP(g,\psi)\}\neq\emptyset$. For f,φ and C, assume that either (B1) *or* (B2) *holds. For g,ψ and K, assume that either* (B1) *or* (B2) *also holds, then the sequences $\{x_n\}$ and $\{u_n\}$ converge weakly to an element $p\in MEP(f,\varphi)$, while $\{w_n\}$ converges weakly to $Ap\in MEP(g,\psi)$.*

If $\varphi=\psi=0$ in Theorem 3.1, then the split generalized mixed equilibrium problem (SGMEP) is reduced to a split generalized equilibrium problem (SGEP).

Corollary 3.2 *Let C be a nonempty closed convex subset of H_1 and K be a nonempty closed convex subset of H_2, where H_1 and H_2 are two real Hilbert spaces. Let $f:C\times C\to R$ and $g:K\times K\to R$ be two bi-functions which satisfy* (A1)-(A4),

$A : H_1 \to H_2$ be a bounded linear operator with the adjoint operator A^*, $B : C \to H_1$ be a α-inverse-strongly monotone mapping and $S : K \to H_2$ be a β-inverse-strongly monotone mapping. Assume that $GEP(f, B) \neq \emptyset$ and $GEP(g, S) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated in the following manner:

$$\begin{cases} x_1 \in C, \\ f(u_n, y) + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle + \langle Bx_n, y - u_n\rangle \geq 0, \quad y \in C, \\ g(w_n, z) + \frac{1}{r}\langle z - w_n, w_n - Au_n\rangle + \langle S(Au_n), z - w_n\rangle \geq 0, \quad z \in K, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)P_C(u_n + \mu A^*(w_n - Au_n)), \quad \text{for any } n \in N, \end{cases}$$

where $r \in (0, a)$, $a = \min\{2\alpha, 2\beta\}$ and $\alpha_n \in (0, 1)$, $\mu \in (0, \frac{1}{\|A^*\|^2})$ are constants. Suppose that $\Omega = \{x \in GEP(f, B) : Ax \in GEP(g, S)\} \neq \emptyset$, then the sequences $\{x_n\}$ and $\{u_n\}$ converge weakly to an element $p \in GEP(f, B)$, while $\{w_n\}$ converges weakly to $Ap \in GEP(g, S)$.

If $B = S = \theta$ and $\varphi = \psi = 0$ in Theorem 3.1, then the split generalized mixed equilibrium problem (SGMEP) is reduced to a split equilibrium problem (SEP) (see [12]).

Corollary 3.3 *Let C be a nonempty closed convex subset of H_1 and K be a nonempty closed convex subset of H_2, where H_1 and H_2 are two real Hilbert spaces. Let $f : C \times C \to R$ and $g : K \times K \to R$ be two bi-functions which satisfy* (A1)-(A4), *$A : H_1 \to H_2$ be a bounded linear operator with the adjoint operator A^*. Assume that $EP(f) \neq \emptyset$ and $EP(g) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated in the following manner:*

$$\begin{cases} x_1 \in C, \\ f(u_n, y) + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad y \in C, \\ g(w_n, z) + \frac{1}{r}\langle z - w_n, w_n - Au_n\rangle \geq 0, \quad z \in K, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)P_C(u_n + \mu A^*(w_n - Au_n)), \quad \text{for any } n \in N, \end{cases}$$

where $r > 0$ and $\alpha_n \in (0, 1)$, $\mu \in (0, \frac{1}{\|A^*\|^2})$ are constants. Suppose that $\Omega = \{x \in EP(f) : Ax \in EP(g)\} \neq \emptyset$, then the sequences $\{x_n\}$ and $\{u_n\}$ converge weakly to an element $p \in EP(f)$, while $\{w_n\}$ converges weakly to $Ap \in EP(g)$.

Theorem 3.2(Strong convergence theorem) *Let C be a nonempty closed convex subset of H_1 and K be a nonempty closed convex subset of H_2, where H_1 and H_2 are two real Hilbert spaces. Let $f : C \times C \to R$ and $g : K \times K \to R$ be two bi-functions which satisfy* (A1)-(A4), *$\varphi : C \to R$ be a lower semicontinuous and convex function, $\psi : K \to R$ be a lower semicontinuous and convex function, $A : H_1 \to H_2$ be a bounded linear operator with the adjoint operator A^*, $B : C \to H_1$ be an α-inverse-strongly monotone mapping and $S : K \to H_2$ be a β-inverse-strongly monotone* *mapping. Assume that $GMEP(f,\varphi,B)\neq\emptyset$ and $GMEP(g,\psi,S)\neq\emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated in the following manner:*

$$\begin{cases} x_1\in C=C_1, \\ f(u_n,y)+\varphi(y)-\varphi(u_n)+\frac{1}{r}\langle y-u_n,u_n-x_n\rangle+\langle Bx_n,y-u_n\rangle\geq 0, \quad y\in C, \\ g(w_n,z)+\psi(z)-\psi(w_n)+\frac{1}{r}\langle z-w_n,w_n-Au_n\rangle+\langle S(Au_n),z-w_n\rangle\geq 0, \quad z\in K, \\ y_n=\alpha_n u_n+(1-\alpha_n)P_C(u_n+\mu A^*(w_n-Au_n)), \quad \text{for any } n\in N, \\ C_{n+1}=\{v\in C_n:\|y_n-v\|\leq\|u_n-v\|\leq\|x_n-v\|\}, \quad \text{for any } n\in N, \\ x_{n+1}=P_{C_{n+1}}(x_0), \end{cases} \tag{3.19}$$

where $r\in(0,a)$, $a=\min\{2\alpha,2\beta\}$ and $\alpha_n\in(0,1)$, $\mu\in(0,\frac{1}{\|A^*\|^2})$ are constants. Suppose that $\Omega\neq\emptyset$. For f,φ and C, assume that either (B1) *or* (B2) *holds. For g,ψ and K, assume that either* (B1) *or* (B2) *also holds, then the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to an element $p\in GMEP(f,\varphi,B)$, while $\{w_n\}$ converges strongly to $Ap\in GMEP(g,\psi,S)$.*

Proof By Lemma 2.4, it follows that

$$\begin{aligned} &GMEP(f,\varphi,B)=Fix(T_r^{f,\varphi}(I-rB)), \quad GMEP(g,\psi,S)=Fix(T_r^{g,\psi}(I-rS)), \\ &u_n=T_r^{f,\varphi}(I-rB)x_n, \quad w_n=T_r^{g,\psi}(I-rS)Au_n. \end{aligned}$$

In fact $\Omega\in C_n$, for $n\in N$. For each $x^*\in\Omega$, it follows from (3.10), (3.11) and (3.6) that

$$\begin{aligned} &\|y_n-x^*\|^2 \\ =&\|\alpha_n u_n+(1-\alpha_n)P_C(u_n+\mu A^*(w_n-Au_n))-x^*\|^2 \\ \leq&\alpha_n\|u_n-x^*\|^2+(1-\alpha_n)\|P_C(u_n+\mu A^*(w_n-Au_n))-P_Cx^*\|^2 \\ \leq&\alpha_n\|u_n-x^*\|^2+(1-\alpha_n)(\|u_n-x^*\|^2 \\ &+\|\mu A^*(w_n-Au_n)\|^2+2\mu\langle u_n-x^*,A^*(w_n-Au_n)\rangle) \\ \leq&\alpha_n\|u_n-x^*\|^2+(1-\alpha_n)(\|u_n-x^*\|^2+\mu^2\|A^*\|^2\|w_n-Au_n\|^2-\mu\|w_n-Au_n\|^2) \\ =&\alpha_n\|u_n-x^*\|^2+(1-\alpha_n)\|u_n-x^*\|^2-\mu(1-\alpha_n)(1-\mu\|A^*\|^2)\|w_n-Au_n\|^2 \\ =&\|u_n-x^*\|^2-\mu(1-\alpha_n)(1-\mu\|A^*\|^2)\|w_n-Au_n\|^2 \\ \leq&\|x_n-x^*\|^2-\mu(1-\alpha_n)(1-\mu\|A^*\|^2)\|w_n-Au_n\|^2. \end{aligned} \tag{3.20}$$

This shows that

$$\|y_n-x^*\|^2\leq\|u_n-x^*\|^2\leq\|x_n-x^*\|^2.$$

It implies that $x^*\in C_{n+1}\subset C_n$, so $\Omega\in C_{n+1}\subset C_n$ and $C_n\neq\emptyset$ for all $n\in N$.

Next we show that C_n is a closed convex set for $n\in N$. It is obvious that C_n is closed for $n\in N$, so we just need to prove that C_n is convex for $n\in N$. In fact, let $v_1,v_2\in C_{n+1}$ for each $\lambda\in(0,1)$, then we have

$$\begin{aligned}
&\|y_n-(\lambda v_1+(1-\lambda)v_2)\|^2\\
&=\|\lambda(y_n-v_1)+(1-\lambda)(y_n-v_2)\|^2\\
&=\lambda\|y_n-v_1\|^2+(1-\lambda)\|y_n-v_2\|^2-\lambda(1-\lambda)\|v_1-v_2\|^2\\
&\le\lambda\|u_n-v_1\|^2+(1-\lambda)\|u_n-v_2\|^2-\lambda(1-\lambda)\|v_1-v_2\|^2\\
&=\|u_n-(\lambda v_1+(1-\lambda)v_2)\|^2. \qquad (3.21)
\end{aligned}$$

namely

$$\|y_n-(\lambda v_1+(1-\lambda)v_2)\|\le\|u_n-(\lambda v_1+(1-\lambda)v_2)\|.$$

In a similar way, we can obtain that $\|u_n-(\lambda v_1+(1-\lambda)v_2)\|\le\|x_n-(\lambda v_1+(1-\lambda)v_2)\|$. This shows $\lambda v_1+(1-\lambda)v_2\in C_{n+1}$, so C_{n+1} is a convex set for $n\in N$.

By (4) of Lemma 2.4, $Fix(T_r^{f,\varphi})$ is closed and convex. Since $T_r^f(I-rB)$ is nonexpansive, we see that $Fix(T_r^{f,\varphi}(I-rB))$ is closed convex. So Ω is a closed convex set, and there exits a unique element $q=P_\Omega(x_0)\in\Omega\subset C_n$. For $x_n=P_{C_n}(x_0)$ and $q\in\Omega\subset C_n$, we get $\|x_n-x_0\|\le\|q-x_0\|$, which implies that $\{x_n\}$ is bounded, so are $\{u_n\}$ and $\{y_n\}$.

Note that $C_{n+1}\subset C_n$ and $x_{n+1}=P_{C_{n+1}}(x_0)\in C_{n+1}$, we obtain that

$$\|x_{n+1}-x_0\|\le\|x_n-x_0\|, \qquad (3.22)$$

which shows that $\lim\limits_{n\to\infty}\|x_n-x_0\|$ exists.

For some $m,n\in N$ with $m>n$, from $x_m=P_{C_m}(x_0)$ and (3) of Lemma 2.1 we arrive at

$$\|x_n-x_m\|^2+\|x_0-x_m\|^2=\|x_n-P_{C_m}(x_0)\|^2+\|x_0-P_{C_m}(x_0)\|^2\le\|x_n-x_0\|^2. \qquad (3.23)$$

Applying (3.22) and (3.23), we see $\lim\limits_{n\to\infty}\|x_n-x_m\|=0$, so $\{x_n\}$ is a Cauchy sequence. Let $x_n\to p$.

Now we prove $p\in\Omega$. Since $x_{n+1}=P_{C_{n+1}}(x_0)\in C_n$, from (3.19) we get

$$\|y_n-x_n\|\le\|y_n-x_{n+1}\|+\|x_n-x_{n+1}\|\le2\|x_n-x_{n+1}\|\to0, \qquad (3.24)$$

$$\|u_n-x_n\|\le\|u_n-x_{n+1}\|+\|x_n-x_{n+1}\|\le2\|x_n-x_{n+1}\|\to0, \qquad (3.25)$$

$$\|y_n-u_n\|\le\|y_n-x_n\|+\|x_n-u_n\|\to0. \qquad (3.26)$$

Using (3.20) and (3.26), we obtain

$$\begin{aligned}
\|w_n-Au_n\|^2&\le\frac{1}{\mu(1-\alpha_n)(1-\mu\|A^*\|^2)}(\|x_n-x^*\|^2-\|y_n-x^*\|^2)\\
&\le\frac{1}{\mu(1-\alpha_n)(1-\mu\|A^*\|^2)}\|x_n-y_n\|(\|x_n-x^*\|+\|y_n-x^*\|)\to0, \qquad (3.27)
\end{aligned}$$

namely

$$\lim_{n\to\infty}\|w_n - Au_n\| = \lim_{n\to\infty}\|T_r^{g,\psi}(I-rS)Au_n - Au_n\| = 0. \tag{3.28}$$

Since $T_r^{f,\varphi}(I-rB)$ is nonexpansive and $x_n \to 0$, it follows from (3.25) that

$$\begin{aligned}\|T_r^{f,\varphi}(I-rB)p-p\| &\le \|T_r^{f,\varphi}(I-rB)p - T_r^{f,\varphi}(I-rB)x_n\| \\ &\quad +\|T_r^{f,\varphi}(I-rB)x_n - x_n\| + \|x_n - p\| \\ &\le \|x_n-p\| + \|u_n - x_n\| + \|x_n - p\| \to 0,\end{aligned}$$

which yields that $p \in GMEP(f,\varphi,B)$. Furthermore we have $\|Ax_n - Ap\| \to 0$ by $x_n \to p$. Then by (3.28), we see that

$$\begin{aligned}&\|T_r^{g,\psi}(I-rS)Ap - Ap\| \\ \le\ &\|T_r^{g,\psi}(I-rS)Ap - T_r^{g,\psi}(I-rS)Ax_n\| + \|T_r^{g,\psi}(I-rS)Ax_n - Ax_n\| + \|Ax_n - Ap\| \\ \le\ &\|Ax_n - Ap\| + \|T_r^{g,\psi}(I-rS)Ax_n - Ax_n\| + \|Ax_n - Ap\| \to 0,\end{aligned}$$

which yields that $Ap \in GMEP(g,\psi,S)$. Hence $\{x_n\}$ converges strongly to $p \in \Omega$ and $\{u_n\}$ converges strongly to $p \in \Omega$ by (3.25).

Then, we get $Au_n \to Ap$ by $u_n \to p$. Note that $\lim\limits_{n\to\infty}\|w_n - Au_n\| = 0$ by (3.28), so $w_n \to Ap$. This completes the proof.

Corollary 3.4 *Let C be a nonempty closed convex subset of H_1 and K be a nonempty closed convex subset of H_2, where H_1 and H_2 are two real Hilbert spaces. Let $f : C\times C \to R$ and $g : K\times K \to R$ be two bi-functions which satisfy* (A1)-(A4), *$\varphi : C \to R$ be a lower semicontinuous and convex function, $\psi : K \to R$ be a lower semicontinuous and convex function and $A : H_1 \to H_2$ be a bounded linear operator with the adjoint operator A^*. Assume that $MEP(f,\varphi) \neq \emptyset$ and $MEP(g,\psi) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated in the following manner:*

$$\begin{cases} x_1 \in C = C_1, \\ f(u_n,y) + \varphi(y) - \varphi(u_n) + \frac{1}{r}\langle y-u_n, u_n - x_n\rangle \ge 0, \quad y \in C, \\ g(w_n,z) + \psi(z) - \psi(w_n) + \frac{1}{r}\langle z - w_n, w_n - Au_n\rangle \ge 0, \quad z \in K, \\ y_n = \alpha_n u_n + (1-\alpha_n)P_C(u_n + \mu A^*(w_n - Au_n)), \quad \text{for any } n \in N, \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \le \|u_n - v\| \le \|x_n - v\|\}, \quad \text{for any } n \in N, \\ x_{n+1} = P_{C_{n+1}}(x_0), \end{cases}$$

where $r > 0$ and $\alpha_n \in (0,1)$, $\mu \in (0, \frac{1}{\|A^*\|^2})$ are constants. Suppose that $\Omega \neq \emptyset$. For f,φ and C, assume that either (B1) or (B2) *holds. For g,ψ and K, assume that either* (B1) or (B2) *also holds, then the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to an element $p \in MEP(f,\varphi)$, while $\{w_n\}$ converges strongly to $Ap \in MEP(g,\psi)$.*

Corollary 3.5 *Let C be a nonempty closed convex subset of H_1 and K be a nonempty closed convex subset of H_2, where H_1 and H_2 are two real Hilbert spaces. Let $f : C\times C \to R$ and $g : K\times K \to R$ be two bi-functions which satisfy* (A1)-(A4),

$A: H_1 \to H_2$ be a bounded linear operator with the adjoint operator A^*, $B: C \to H_1$ be an α-inverse-strongly monotone mapping and $S: K \to H_2$ be a β-inverse-strongly monotone mapping. Assume that $GEP(f,B) \neq \emptyset$ and $GEP(g,S) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated in the following manner:

$$\begin{cases} x_1 \in C = C_1, \\ f(u_n,y) + \frac{1}{r}\langle y-u_n, u_n-x_n\rangle + \langle Bx_n, y-u_n\rangle \geq 0, \quad y \in C, \\ g(w_n,z) + \frac{1}{r}\langle z-w_n, w_n-Au_n\rangle + \langle S(Au_n), z-w_n\rangle \geq 0, \quad z \in K, \\ y_n = \alpha_n u_n + (1-\alpha_n)P_C(u_n + \mu A^*(w_n - Au_n)), \quad \text{for any } n \in N, \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|u_n - v\| \leq \|x_n - v\|\}, \quad \text{for any } n \in N, \\ x_{n+1} = P_{C_{n+1}}(x_0), \end{cases}$$

where $r \in (0,a), a = \min\{2\alpha, 2\beta\}$ and $\alpha_n \in (0,1)$, $\mu \in (0, \frac{1}{\|A^*\|^2})$ are constants. Suppose that $\Omega = \{x \in GEP(f,B) : Ax \in GEP(g,S)\} \neq \emptyset$, then the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to an element $p \in GEP(f,B)$, while $\{w_n\}$ converges strongly to $Ap \in GEP(g,S)$.

Corollary 3.6 *Let C be a nonempty closed convex subset of H_1 and K be a nonempty closed convex subset of H_2, where H_1 and H_2 are two real Hilbert spaces. Let $f: C \times C \to R$ and $g: K \times K \to R$ be two bi-functions which satisfy* (A1)-(A4) *and $A: H_1 \to H_2$ be a bounded linear operator with the adjoint operator A^*. Assume that $EP(f) \neq \emptyset$ and $EP(g) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated in the following manner:*

$$\begin{cases} x_1 \in C = C_1, \\ f(u_n,y) + \frac{1}{r}\langle y-u_n, u_n-x_n\rangle \geq 0, \quad y \in C, \\ g(w_n,z) + \frac{1}{r}\langle z-w_n, w_n-Au_n\rangle \geq 0, \quad z \in K, \\ y_n = \alpha_n u_n + (1-\alpha_n)P_C(u_n + \mu A^*(w_n - Au_n)), \quad \text{for any } n \in N, \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|u_n - v\| \leq \|x_n - v\|\}, \quad \text{for any } n \in N, \\ x_{n+1} = P_{C_{n+1}}(x_0), \end{cases}$$

where $r > 0$ and $\alpha_n \in (0,1)$, $\mu \in (0, \frac{1}{\|A^*\|^2})$ are constants. Suppose that $\Omega = \{x \in EP(f) : Ax \in EP(g)\} \neq \emptyset$, then the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to an element $p \in EP(f)$, while $\{w_n\}$ converges strongly to $Ap \in EP(g)$.

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