Ann. of Appl. Math.
32:1(2016), 69-78

CYCLES EMBEDDING ON FOLDED HYPERCUBES WITH FAULTY NODES*†

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Abstract

Let FF_v be the set of faulty nodes in an n-dimensional folded hypercube FQ_n with $|FF_v| \leq n-1$ and all faulty vertices are not adjacent to the same vertex. In this paper, we show that if $n \geq 4$, then every edge of $FQ_n - FF_v$ lies on a fault-free cycle of every even length from 6 to $2^n - 2|FF_v|$.

Keywords folded hypercube; interconnection network; fault-tolerant; path

2000 Mathematics Subject Classification 68M15; 68M10

1 Introduction

The n-dimensional hypercube Q_n (or n-cube) is one of the most important topology of networks due to its excellent properties such as regularity, recursive structure, small diameter, vertex and edge transitive and relatively short mean distance [1]. In order to improve the performance of hypercube, the folded hypercube FQ_n has been proposed [2].

Since a large-scale hypercube network fails in any component, it's desirable that the rest of the network continue to operate in spite of the failure. This leads to the graph-embedding problem with faulty edges and/or vertices. This problem has received much attention (see [3-10]).

The problem of embedding paths in an n-dimensional hypercube and folded hypercube has been well studied. Tsai [3] showed that for any subset F_v of $V(Q_n)$ with $|F_v| \leq n-2$, every edge of $Q_n - F_v$ lies on a cycle of every even length from 4 to $2^n - 2|F_v|$ inclusive. Tsai [4] also showed that for any subset F_v of $V(Q_n)$ with $|F_v| \leq n-1$ and all faulty vertices are not adjacent to the same vertex, every edge of $Q_n - F_v$ lies on a cycle of every even length from 6 to $2^n - 2|F_v|$ inclusive. Hsieh

*This project was supported by NSFC (11371162) and NSFC(11171129) and HuBei (T201103).

†Manuscript received October 10, 2015

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and Shen [5] proved that every edge of $Q_n - F_v - F_e$ lies on a cycle of every even length from 4 to $2^n - 2|F_v|$ even if $|F_v| + |F_e| \leq n-2$, where $n \geq 3$.

Let FF_v and FF_e denote the set of faulty nodes and faulty edges of FQ_n respectively. Hsieh, Kuo and Huang [6] proved that if the folded hypercube FQ_n has just only one fault node, then FQ_n contains cycles of every even length from 4 to 2^n-2 if $n \geq 3$, and cycles of every odd length from $n+1$ to 2^n-1 when n is even, $n \geq 2$. Ma, Xu and Du [7] further demonstrated that $FQ_n - FF_e$ ($n \geq 3$) with $|FF_e| \leq 2n-3$ contains a fault-free cycle passing through all nodes if each vertex is incident with at least two fault-free edges. Kuo and Hsieh [8] improved the conclusion of [7] and proved that $FQ_n - FF_e$ with $|FF_e| = 2n-3$ contains a fault-free cycle of every even length from 4 to 2^n. Xu, Ma and Du [9] further showed that every fault-free edge of $FQ_n - FF_e$ lies on a fault-free cycle of every even length from 4 to 2^n and every odd length from $n+1$ to 2^n-1 if n is even, where $|FF_e| \leq n-1$. Then Cheng, Hao and Feng [10] proved that every fault-free edge of $FQ_n - FF_v$ lies on a fault-free cycle of every even length from 4 to $2^n - 2|FF_v|$ and every odd length from $n+1$ to $2^n - 2|FF_v| - 1$ if n is even, where $|FF_v| \leq n-2$.

In this paper, under the conditional $|FF_v| \leq n-1$ and all faulty vertices are not adjacent to the same vertex, we show that if $n \geq 4$, then every edge of $FQ_n - FF_v$ lies on a fault-free cycle of every even length from 6 to $2^n - 2|FF_v|$.

2 Preliminaries

Please see [1] for graph-theoretical terminology and notation is not defined here. A network is usually modeled by a simple connected graph $G = (V, E)$, where $V = V(G)$ (or $E = E(G)$) is the set of vertices (or edges) of G. We define the vertex x to be a neighbor of y if $xy \in E(G)$. A graph G is bipartite if X, Y are two disjoint subsets of $V(G)$ such that $E(G) = \{xy|\, x \in X, y \in Y\}$. A graph $P = (u_1, u_2, \cdots, u_k)$ is called a path if the vertices $u_1, u_2, \cdots, u_k$ are distinct and any two consecutive vertices u_i and u_{i+1} are adjacent. u_1 and u_k are called the end-vertices of P. If $u_1 = u_k$, the path $P(u_1, u_k)$ is called a cycle (denoted by C). The length of a path P (a cycle C), denoted by $l(P)$ (or $l(C)$), is the number of edges in P (or C). In general, the distance of two vertices x, y is the length of the shortest (x, y)-path.

The n-dimensional hypercube Q_n (or, n-cube) can be represented as an undirected graph with 2^n vertices. Every vertex $x \in Q_n$ is labeled as a binary string $x_1x_2\cdots x_n$ of length n from $00\cdots0$ to $11\cdots1$. Two vertices u and v are adjacent if their binary strings differ in exactly one bit. For convenience, we call $e \in E$ an edge of dimension i if its end-vertices strings differ in ith-bit. In the rest of this paper, we denote $x^i = x_1x_2\cdots\overline{x_i}\cdots x_n$, where $\overline{x_i} = 1 - x_i$, $x_i = 0, 1$. The Hamming distance of two vertices $x = x_1x_2\cdots x_n$ and $y = y_1y_2\cdots y_n$ is $H(x,y) = \sum_{i=1}^{n} |x_i - y_i|$, the number of different bits between them. Let $d_H(x,y)$ be the shortest distance of x and y. Note that Q_n is a bipartite graph, and for any two distinct vertices x, y of Q_n, $d_H(x,y) = H(x,y)$. $N(x)$ denotes a set of the nodes which are neighbors of x.

As a variant of hypercube, the n-dimensional folded hypercube FQ_n is obtained by adding more edges between its vertices.

Definition 1 The n-dimensional folded hypercube FQ_n is a graph with $V(Q_n) = V(FQ_n)$. Two vertices $x = x_1x_2\cdots x_n$ and y are connected by an edge if and only if

(i) $y = x_1x_2\cdots\overline{x_i}\cdots x_n$ (denoted by x^i), or

(ii) $y = \overline{x_1}\overline{x_2}\cdots\overline{x_i}\cdots\overline{x_n}$ (denoted by $\overline{x}$).

Therefore, the hypercube Q_n is a spanning subgraph of the folded hypercube FQ_n obtained by removing the second type of edges $x\overline{x}$ ($x \in V(FQ_n)$), called complementary edges of FQ_n and denoted by $E_c = \{x\overline{x} |\, x \in V(FQ_n)\}$.

In general, the first type of edges are defined to be the hypercube edges, and denoted by $E_i = \{xx^i\}$, $i = 1, 2, \cdots, n$.

Lemma 1 *An i-partition on FQ_n, where $1 \le i \le n$, is a partition of FQ_n along dimension i into two $n-1$-cubes, denoted by Q^0_{n-1} and Q^1_{n-1}.*

The nodes in Q^0_{n-1} (respectively, Q^1_{n-1}) can also be denoted by $0x$ (respectively, $1x$) for brevity, where satisfying $0x = x_1x_2\cdots x_i\cdots x_n \in Q^0_{n-1}$ satisfying $x_i = 0$ (respectively, $1x = x_1x_2\cdots x_i\cdots x_n \in Q^1_{n-1}$ satisfying $x_i = 1$).

Lemma 2[4] *Let $f_e = 0$, $f_v = n-1$, and every fault-free vertex is adjacent to at least two fault-free vertices in Q_n for $n \ge 4$. Then, every fault-free edge of Q_n lies on a fault-free cycle of every even length from 6 to $2^n - 2f_v$ inclusive.*

Lemma 3[3] *Assume F_v is any subset of $V(Q_n)$. Every edge in $Q_n - F_v$ lies on a fault-free cycle of every even length from 4 to $2^n - 2f_v$ inclusive even if $|F_v| \le n-2$, where $n \ge 3$.*

Lemma 4[12] *Let $n \ge 2$ be an integer. For any two different fault-free vertices u and v in Q_n with $f_e + f_v \le n-2$, there exists a fault-free uv-path of length l for each l satisfying $d_H(u,v) + 2 \le l \le 2^n - 2f_v - 1$ and $2|(l - d_H(u,v))$. Moreover, there must exist a fault-free uv-path of length $d_H(u,v)$ if $d_H(u,v) \ge n-1$.*

Lemma 5[10] *Assume that FQ_n is partitioned along dimension i ($1 \le i \le n$) into two $n-1$-cubes, denoted by Q^0_{n-1} and Q^1_{n-1}, $0u$ and $0v$ (respectively, $1u$ and $1v$) are two nodes in Q^0_{n-1} (respectively, Q^1_{n-1}). If $d_H(0u, 0v) = n-2$ (respectively, $d_H(1u, 1v) = n-2$), then $d_H(1\overline{u}, 1v) = 1$ and $d_H(1u, 1\overline{v}) = 1$ (respectively, $d_H(0\overline{u}, 0v) = 1$ and $d_H(0u, 0\overline{v}) = 1$); if $d_H(0u, 0v) = 1$ (respectively, $d_H(1u, 1v) = 1$), then $d_H(1\overline{u}, 1v) = n-2$ and $d_H(1u, 1\overline{v}) = n-2$ (respectively,*

$d_H(0\overline{u}, 0v) = n-2$ and $d_H(0u, 0\overline{v}) = n-2$).

Lemma 6[5] *There exists a path of every odd length from 3 to $2^n - 2|F_v| - 1$ joining any two adjacent fault-free nodes in $Q_n - F_v$ even if $|F_e| = 0$ and $|F_v| \leq n-2$, where $n \geq 3$.*

Lemma 7[10] *Assume n is even and FF_v is any subset of $V(FQ_n)$. Every edge of $FQ_n - FF_v$ lies on a fault-free cycle of every odd length from $n+1$ to $2^n - 2|FF_v| - 1$ inclusive even if $|FF_v| \leq n-2$, where $n \geq 2$.*

Lemma 8 *Assume that FQ_n is partitioned along dimension i $(1 \leq i \leq n)$ into two $n-1$-cubes, denoted by Q^0_{n-1} and Q^1_{n-1}, $0u$ and $0v$ (respectively, $1u$ and $1v$) are two nodes in Q^0_{n-1} (respectively, Q^1_{n-1}). If $d_H(0u, 0v) = n-3$ (respectively, $d_H(1u, 1v) = n-3$), then $d_H(1\overline{u}, 1v) = 2$ and $d_H(1u, 1\overline{v}) = 2$ (respectively, $d_H(0\overline{u}, 0v) = 2$ and $d_H(0u, 0\overline{v}) = 2$); if $d_H(0u, 0v) = 2$ (respectively, $d_H(1u, 1v) = 2$), then $d_H(1\overline{u}, 1v) = n-3$ and $d_H(1u, 1\overline{v}) = n-3$ (respectively, $d_H(0\overline{u}, 0v) = n-3$ and $d_H(0u, 0\overline{v}) = n-3$).*

Proof If $d_H(0u, 0v) = n-3$, then $d_H(u, v) = n-3$, which implies $d_H(\overline{u}, v) = 2$ and $d_H(u, \overline{v}) = 2$, thus $d_H(1\overline{u}, 1v) = 2$ and $d_H(1u, 1\overline{v}) = 2$. By the similar discussion, if $d_H(1u, 1v) = n-3$, then $d_H(0\overline{u}, 0v) = 2$ and $d_H(0u, 0\overline{v}) = 2$.

If $d_H(0u, 0v) = 2$, then $d_H(u, v) = 2$, which implies $d_H(\overline{u}, v) = n-3$ and $d_H(u, \overline{v}) = n-3$, thus $d_H(1\overline{u}, 1v) = n-3$ and $d_H(1u, 1\overline{v}) = n-3$. By the similar discussion, if $d_H(1u, 1v) = 2$, then $d_H(0\overline{u}, 0v) = n-3$ and $d_H(0u, 0\overline{v}) = n-3$. The proof is completed.

Lemma 9[2] *For any two vertices $u, v \in Q_n$, if $d(u, v) = k$, then there are n internal disjoint paths from u and v such that there are k paths of length k and $n-k$ paths of length $k+2$.*

Lemma 10[10] *Assume FF_v is any subset of $V(FQ_n)$. Every edge in $FQ_n - FF_v$ lies on a fault-free cycle of every even length from 4 to $2^n - 2|FF_v|$ inclusive even if $|FF_v| \leq n-2$, where $n \geq 3$.*

Lemma 11[9] *There is an automorphism σ of FQ_n such that $\sigma(E_i) = E_j$ for any $i, j \in \{1, 2, \cdots, n, c\}$.*

3 Main Results

Before the proof, I give some symbols. FF_v is the set of faulty vertices in FQ_n and FF^i_v is the set of faulty vertices in Q^i_{n-1}, $i = \{0, 1\}$.

Lemma 12 *Assume FF_v is any subset of $V(FQ_4)$. Every edge in $FQ_4 - FF_v$ lies on a fault-free cycle of every even length from 6 to $2^4 - 2|FF_v|$ inclusive even if $|FF_v| \leq 3$ and all faulty vertices are not adjacent to the same vertex.*

Proof If $|FF_v| = f_v \leq 2$, by Lemma 10, the lemma holds. Therefore, we only need to consider the situation of $f_v = 3$, every edge in $FQ_4 - FF_v$ lies on a fault-free cycle of every even length from 6 to 10 inclusive. By Lemma 1, FQ_4 can be partitioned along dimension i into two 3-cubes, denoted by Q_3^0 and Q_3^1. There must exist an i such that $FF_v^0 \not\subseteq N(u)$, $u \in Q_3^0$ and $FF_v^1 \not\subseteq N(v)$, $v \in Q_3^1$ (We can simply divide one of the faulty vertex and the other faulty vertices into different parts (Q_3^0 or Q_3^1) along an i-dimension. The proof is the condition that all faulty vertices are not adjacent to the same vertex. We can consider extreme situation. If $n-2$ faulty vertices are adjacent to the same vertex x, we can choose one of $n-2$ faulty vertices, denoted by y, then x and y have one bit differently. So we can partition along this dimension. Therefore y is in a part, other faulty vertices is in another part and all faulty vertices are not adjacent to the same vertex in this part).

Let $f_v^i = |FF_v \cap Q_3^i|$, $i = 0, 1$, $f_v = f_v^0 + f_v^1 = 3$. Without loss of generality, let $FF_v = \{w_1, w_2, w_3\}$, $FF_v^0 = \{w_1, w_2\} \in Q_3^0$, $FF_v^1 = \{w_3\} \in Q_3^1$. $f_v^0 = 2$, $f_v^1 = 1$. e is a fault-free edge. $f_v^0 = 2$, $FF_v^0 \not\subseteq N(u)$, $u \in Q_3^0$, so $d_H(w_1, w_2) = 1$ or $d_H(w_1, w_2) = 3$.

(1) $e \in Q_3^0$.

Case 1 $d_H(w_1, w_2) = 1$.

Then, $e \in C_4$, that is there exists a cycle C_0 of every even length l_0 containing e in Q_3^0, where $l_0 = 4$. Let $(x, y) \neq e$ be a fault-free edge in cycle C_0 such that (x^i, y^i) (or $(\overline{x}, \overline{y})$) is fault-free in Q_3^1. Let $C_0 = \langle x, P_0, y, x\rangle$, then $l_0' = l(P_0) = 3$. Since $f_v^1 = 1$, by Lemma 4, there exists a path P_1 of every odd length l_1 joining x^i and y^i (or $\overline{x}$ and $\overline{y}$) in Q_3^1, where $3 \le l_1 \le 5$. (x^i, y^i) (or $(\overline{x}, \overline{y})$) is fault-free, there exists a path P_1' of every odd length joining x^i and y^i (or $\overline{x}$ and $\overline{y}$) in Q_3^1, where $1 \le l_1' \le 5$. Let $C = \langle x, P_0, y, y^i, P_1', x^i, x\rangle$ or $C = \langle x, P_0, y, \overline{y}, P_1', \overline{x}, x\rangle$ with even length $l = l_0' + l_1' + 2$. Since $l_0' = 3$ and $1 \le l_1' \le 5$, $6 \le l \le 10$.

Case 2 $d_H(w_1, w_2) = 3$.

Through observation, $e \in C_6$. Let $(x, y) \neq e$ be a fault-free edge in cycle C_0 such that (x^i, y^i) (or $(\overline{x}, \overline{y})$) is fault-free in Q_3^1. Let $C_0 = \langle x, P_0, y, x\rangle$, then $l_0' = l(P_0)$, $l_0' = 5$. Since $f_v^1 = 1$, by Lemma 4, there exists a path P_1 of every odd length l_1 joining x^i and y^i (or $\overline{x}$ and $\overline{y}$) in Q_3^1, where $3 \le l_1 \le 5$. (x^i, y^i) (or $(\overline{x}, \overline{y})$) is fault-free, there exists a path P_1' of every odd length joining x^i and y^i (or $\overline{x}$ and $\overline{y}$) in Q_3^1, where $1 \le l_1' \le 5$. Let $C = \langle x, P_0, y, y^i, P_1', x^i, x\rangle$ or $C = \langle x, P_0, y, \overline{y}, P_1', \overline{x}, x\rangle$ with even length $l = l_0' + l_1' + 2$. Since $l_0' = 5$ and $1 \le l_1' \le 5$, $8 \le l \le 12$. We can obtain the desired even cycle of length 6 in C_0, where $l_0 = 6$. So $6 \le l \le 12$.

(2) $e \in Q_3^1$.

Case 1 $d_H(w_1, w_2) = 1$.

Since $f_v^1 = 1$, by Lemma 3, there exists a cycle C_1 of every even length l_1 containing e in Q_3^1, where $4 \le l_1 \le 6$. Let $(x, y) \neq e$ be a fault-free edge in cycle C_1 such that (x^i, y^i) (or $(\overline{x}, \overline{y})$) is fault-free in Q_3^0. Hence, there exists a path P_1 of every odd length l_1' joining x and y in Q_3^1, where $3 \le l_1' \le 5$. We can choose (x^i, y^i). Since $d_H(w_1, w_2) = 1$, $(x^i, y^i) \in C_4$. $(x^i, y^i) \in C_4$, (x^i, y^i) is fault-free, then there exists a path P_0 of every odd length l_0 joining x^i and y^i, where $1 \le l_0 \le 3$. Let $C = \langle x, P_1, y, y^i, P_0, x^i, x\rangle$ with even length $l = l_0 + l_1' + 2$. Since $1 \le l_0 \le 3$ and $3 \le l_1' \le 5$, $6 \le l \le 10$.

Case 2 $d_H(w_1, w_2) = 3$.

Since $f_v^1 = 1$, by Lemma 3, there exists a cycle C_1 of every even length l_1 containing e in Q_3^1, where $4 \le l_1 \le 6$. Let $(x, y) \ne e$ be a fault-free edge in cycle C_1 such that (x^i, y^i) (or $(\overline{x}, \overline{y})$) is fault-free in Q_3^0. Hence, there exists a path P_1 of every odd length l_1' joining x and y in Q_3^1, where $3 \le l_1' \le 5$. $d_H(w_1, w_2) = 3$, through observation, $(x^i, y^i) \in C_6$ (or $(\overline{x}, \overline{y}) \in C_6$). We can choose (x^i, y^i), then, there exists a path P_0 of every odd length l_0 joining x^i and y^i in Q_3^0, where $l_0 = 5$. Let $C = \langle x, P_1, y, y^i, P_0, x^i, x\rangle$ with even length $l = l_0 + l_1' + 2$. Since $l_0 = 5$ and $3 \le l_1' \le 5$, $10 \le l \le 12$. Let $C = \langle x, P_1, y, y^i, x^i, x\rangle$ with even length $l = 1 + l_1' + 2$, where $3 \le l_1' \le 5$. Then $6 \le l \le 8$. So $6 \le l \le 12$.

(3) $e \in E_i$.

Case 1 $d_H(w_1, w_2) = 1$. Let $e = (x, x^i)$, $x \in Q_3^0$, $x^i \in Q_3^1$.

Let (x, y) be a fault-free edge in such that (x^i, y^i) is fault-free in Q_3^1.

$(x, y) \in C_4$, (x, y) is a fault-free edge, there exists a path P_0 of every odd length l_0 joining x and y in Q_3^0, where $1 \le l_0 \le 3$. Since $f_v^1 = 1$, by Lemma 4, there exists a path P_1 of every odd length l_1 joining x^i and y^i in Q_3^1, where $3 \le l_1 \le 5$. Let $C = \langle x, P_0, y, y^i, P_1, x^i, x\rangle$ with even length $l = l_0 + l_1 + 2$. Since $1 \le l_0 \le 3$ and $3 \le l_1 \le 5$, $6 \le l \le 10$.

Case 2 $d_H(w_1, w_2) = 3$. Let $e = (x, x^i)$, $x \in Q_3^0$, $x^i \in Q_3^1$.

Let (x, y) be a fault-free edge in such that (x^i, y^i) is fault-free in Q_3^1. Through observation, $(x, y) \in C_6$, there exists a path P_0 of every odd length l_0 joining x and y in Q_3^0, where $l_0 = 5$. Since $f_v^1 = 1$, by Lemma 4, there exists a path P_1 of every odd length l_1 joining x^i and y^i in Q_3^1, where $3 \le l_1 \le 5$. Let $C = \langle x, P_0, y, y^i, P_1, x^i, x\rangle$ with even length $l = l_0 + l_1 + 2$. Since $l_0 = 5$ and $3 \le l_1 \le 5$, $10 \le l \le 12$. Let $C = \langle x, y, y^i, P_1, x^i, x\rangle$ with even length $l = 1 + l_1 + 2$. Since $3 \le l_1 \le 5$, $6 \le l \le 8$. Therefore, $6 \le l \le 12$.

(4) $e \in E_c$. Let $e = (x, \overline{x})$, $x \in Q_3^0$, $\overline{x} \in Q_3^1$.

Let $\{\overline{x}, \overline{y}\}$ replace $\{x^i, y^i\}$, the following proof is similar to (3) $e \in E_i$. The proof is completed.

Theorem 1 *Assume FF_v is any subset of $V(FQ_n)$. Every edge in $FQ_n - FF_v$ lies on a fault-free cycle of every even length from 6 to $2^n - 2|FF_v|$ inclusive even if $|FF_v| \le n - 1$ and all faulty vertices are not adjacent to the same vertex, where $n \ge 4$.*

Proof If $|FF_v| = f_v \leq n-2$, by Lemma 10, the theorem holds. When $n=4$, Lemma 12 holds. Therefore, we only need to consider the situation of $|FF_v| = f_v = n-1$, where $n \geq 5$. By Lemma 1, FQ_n can be partitioned along dimension i into two $n-1$-cubes, denoted by Q^0_{n-1} and Q^1_{n-1}. There must exist an i such that $FF^0_v \not\subseteq N(u)$, $u \in Q^0_{n-1}$ and $FF^1_v \not\subseteq N(v)$, $v \in Q^1_{n-1}$ (We can simply divide one of the faulty vertex and the other faulty vertices into different parts (Q^0_{n-1} or Q^1_{n-1}) along an i-dimension. The proof is the condition that all faulty vertices are not adjacent to the same vertex. We can consider extreme situation. If $n-2$ faulty vertices are adjacent to the same vertex x, we can choose one of $n-2$ faulty vertices, denoted by y, then x and y have one bit differently. So we can partition along this dimension. Therefore y is in a part, other faulty vertices is in another part and all faulty vertices are not adjacent to the same vertex in this part).

Let $f^i_v = |FF_v \cap Q^i_{n-1}|$, $i=0,1$, $f_v = f^0_v + f^1_v = n-1$. e is a fault-free edge.

<u>Case 1</u> If there exists an $i \in \{1,2,\cdots,n\}$ such that $f^0_v = n-2$, $f^1_v = 1$, $FQ_n = Q^0_{n-1} \cup Q^1_{n-1}$, $FF^0_v \not\subseteq N(u)$, $u \in Q^0_{n-1}$.

Case 1.1 $e \in Q^0_{n-1}$.

Since $f^0_v = n-2$, by Lemma 2, there exists a cycle C_0 of every even length l_0 containing e in Q^0_{n-1}, where $6 \leq l_0 \leq 2^{n-1} - 2f^0_v$. Let $(x,y) \neq e$ be a fault-free edge in cycle C_0 such that (x^i, y^i) (or $(\overline{x}, \overline{y})$) is fault-free in Q^1_{n-1} (Since $f^1_v = 1$). Let $C_0 = \langle x, P_0, y, x\rangle$, then $l'_0 = l(P_0)$, $5 \leq l'_0 \leq 2^{n-1} - 2f^0_v - 1$. Since $f^1_v = 1$, by Lemma 3, there exists a cycle C_1 of even length l_1 containing edge (x^i, y^i) (or $(\overline{x}, \overline{y})$) in Q^1_{n-1}, where $4 \leq l_1 \leq 2^{n-1} - 2f^1_v$. Hence, there exists a path P_1 of odd length l'_1 joining x^i and y^i (or $\overline{x}$ and $\overline{y}$), where $3 \leq l'_1 \leq 2^{n-1} - 2f^1_v - 1$. Let $C = \langle x, P_0, y, y^i, P_1, x^i, x\rangle$ or $C = \langle x, P_0, y, \overline{y}, P_1, \overline{x}, x\rangle$ with even length $l = l'_0 + l'_1 + 2$. Since $5 \leq l'_0 \leq 2^{n-1} - 2f^0_v - 1$ and $3 \leq l'_1 \leq 2^{n-1} - 2f^1_v - 1$, $10 \leq l \leq 2^n - 2(f^0_v + f^1_v)$. We can obtain the desired even cycle of length from 6 to 8 in C_0, where $6 \leq l_0 \leq 2^{n-1} - 2f^0_v$. So $6 \leq l \leq 2^n - 2(f^0_v + f^1_v)$.

Case 1.2 $e \in Q^1_{n-1}$.

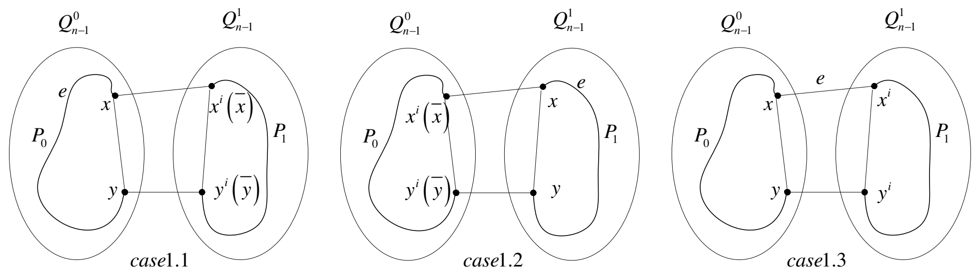
Since $f^1_v = 1$, by Lemma 3, there exists a cycle C_1 of even length l_1 containing edge e in Q^1_{n-1}, where $4 \leq l_1 \leq 2^{n-1} - 2f^1_v$. Let C_k be a fault-free k-cycle covering the edge e in Q^1_{n-1}, where $k = 2^{n-1} - 2f^1_v$. Obviously, there are $2^{n-2} - f^1_v$ mutually disjoint edges excluding e in C_k. $2(2^{n-2} - f^1_v) \geq f^0_v$ is easy to be hold, where $f^0_v = n-2, f^1_v = 1$. Thus, there exists an $(x,y) \neq e$ which is a fault-free edge in cycle C_1 such that (x^i, y^i) (or $(\overline{x}, \overline{y})$) is fault-free in Q^0_{n-1}. Let $C_1 = \langle x, P_1, y, x\rangle$, then $l'_1 = l(P_1)$, $3 \leq l'_1 \leq 2^{n-1} - 2f^1_v - 1$. Since $f^0_v = n-2$, and (x^i, y^i) (or $(\overline{x}, \overline{y})$) is fault-free edge, by Lemma 2, there exists a cycle C_0 of even length l_0 containing edge (x^i, y^i) (or $(\overline{x}, \overline{y})$) in Q^0_{n-1}, where $6 \leq l_0 \leq 2^{n-1} - 2f^0_v$. Hence, there exists a path P_0 of odd length l'_0 joining x^i and y^i (or $\overline{x}$ and $\overline{y}$), where $5 \leq l'_0 \leq 2^{n-1} - 2f^0_v - 1$. Let

$C = \langle x, P_1, y, y^i, P_0, x^i, x\rangle$ or $C = \langle x, P_1, y, \overline{y}, P_0, \overline{x}, x\rangle$ with even length $l = l_0'+l_1'+2$. Since $5 \le l_0' \le 2^{n-1} - 2f_v^0 - 1$ and $3 \le l_1' \le 2^{n-1} - 2f_v^1 - 1$, $10 \le l \le 2^n - 2(f_v^0 + f_v^1)$. We can obtain the desired even cycle of length from 6 to 8 in C_1, where $4 \le l_1 \le 2^{n-1} - 2f_v^1$. So $6 \le l \le 2^n - 2(f_v^0 + f_v^1)$.

Case 1.3 $e \in E_i$.

Let $e = (x, x^i)$, $x \in Q_{n-1}^0$, $x^i \in Q_{n-1}^1$.

Since $f_v^0 = n-2$, $f_v^1 = 1$, $FF_v^0 \not\subseteq N(u)$, $u \in Q_{n-1}^0$, x has at least 2 fault-free neighbors y_1, y_2 in Q_{n-1}^0. $f_v^1 = 1$, one of the y_1^i, y_2^i must be fault-free in Q_{n-1}^1. Therefore, there must exist an edge (x, y) in Q_{n-1}^0 such that (x^i, y^i) is fault-free in Q_{n-1}^1. Since $f_v^0 = n-2$, by Lemma 2, there exists a cycle C_0 of every even length l_0 containing (x, y) in Q_{n-1}^0, where $6 \le l_0 \le 2^{n-1} - 2f_v^0$. Let $C_0 = \langle x, P_0, y, x\rangle$, then $l_0' = l(P_0)$, $5 \le l_0' \le 2^{n-1} - 2f_v^0 - 1$. Since $f_v^1 = 1$, by Lemma 6, there exists a cycle P_1 of odd length l_1 joining x^i and y^i, where $3 \le l_1 \le 2^{n-1} - 2f_v^1 - 1$. Since (x^i, y^i) is fault-free, there exists a cycle P_1' of odd length l_1' joining x^i and y^i, where $1 \le l_1' \le 2^{n-1} - 2f_v^1 - 1$. Let $C = \langle x, P_0, y, y^i, P_1', x^i, x\rangle$ with even length $l = l_0'+l_1'+2$. Since $5 \le l_0' \le 2^{n-1} - 2f_v^0 - 1$ and $1 \le l_1' \le 2^{n-1} - 2f_v^1 - 1$, $8 \le l \le 2^n - 2(f_v^0 + f_v^1)$. Let $C = \langle x, y, y^i, P_1, x^i, x\rangle$ with $l = 1 + l(P_1) + 2$, $l(P_1) = 3$, we can obtain the desired even cycle of length 6. So $6 \le l \le 2^n - 2(f_v^0 + f_v^1)$.



Case 1.4 $e \in E_c$.

The following proof is similar to Case 1.3.

Case 2 If there exists an $i \in \{1, 2, \cdots, n\}$ such that $f_v^0 \le f_v^1 \le n-3$. $FQ_n = Q_{n-1}^0 \cup Q_{n-1}^1$.

Case 2.1 $e \in Q_{n-1}^0$.

Since $f_v^0 \le n-3$, by Lemma 3, there exists a cycle C_0 of every even length l_0 containing edge e in Q_{n-1}^0, where $4 \le l_0 \le 2^{n-1} - 2f_v^0$. Let C_k be a fault-free k-cycle covering the edge e in Q_{n-1}^0, where $k = 2^{n-1} - 2f_v^0$. Obviously, there are $2^{n-2} - f_v^0$ mutually disjoint edges excluding e in C_k. $2(2^{n-2} - f_v^0) > f_v^1$ is easy to be hold, where $f_v^0 \le f_v^1 \le n-3$. Thus, there exists an $(x, y) \ne e$ which is a fault-free edge in cycle C_k such that (x^i, y^i) (or $(\overline{x}, \overline{y})$) is fault-free in Q_{n-1}^1. Then, there exists a path

P_0 of every odd length l'_0 joining x and y in Q^0_{n-1}, where $3 \leq l'_0 \leq 2^{n-1} - 2f^0_v - 1$. Since $f^1_v \leq n-3$, by Lemma 3, there exists a cycle C_1 of every even length l_1 containing edge (x^i, y^i) (or $(\overline{x}, \overline{y})$) in Q^1_{n-1}, where $4 \leq l_1 \leq 2^{n-1} - 2f^1_v$. (x^i, y^i) (or $(\overline{x}, \overline{y})$) is fault-free edge, so there exists a path P_1 of odd length l'_1 joining x^i and y^i (or $\overline{x}$ and $\overline{y}$), where $1 \leq l'_1 \leq 2^{n-1} - 2f^1_v - 1$. Let $C = \langle x, P_0, y, y^i, P_1, x^i, x\rangle$ or $C = \langle x, P_0, y, \overline{y}, P_1, \overline{x}, x\rangle$ with even length $l = l'_0 + l'_1 + 2$. Since $3 \leq l'_0 \leq 2^{n-1} - 2f^0_v - 1$ and $1 \leq l'_1 \leq 2^{n-1} - 2f^1_v - 1$, $6 \leq l \leq 2^n - 2(f^0_v + f^1_v)$.

Case 2.2 $e \in Q^1_{n-1}$.

The following proof is similar to Case 2.1.

Case 2.3 $e \in E_i$.

By Lemma 11, the proof is completed.

Case 2.4 $e \in E_c$.

By Lemma 11, the proof is completed.

The proof of Theorem 1 is finished.

4 Conclusion

The folded hypercube FQ_n is an important network topology for parallel processing computer systems. According to [4], we can prove the same conclusion in FQ_n. Under the condition $|FF_v| \leq n-1$ and all faulty vertices are not adjacent to the same vertex, we show that if $n \geq 4$, then every edge of $FQ_n - FF_v$ lies on a fault-free cycle of every even length from 6 to $2^n - 2|FF_v|$.

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(*edited by Mengxin He*)