Ann. of Appl. Math.
34:1(2018), 32-46

ALMOST PERIODIC SOLUTION OF A NONAUTONOMOUS MODIFIED LESLIE-GOWER PREDATOR-PREY MODEL WITH NONMONOTONIC FUNCTIONAL RESPONSE AND A PREY REFUGE*†

Jinhuang Chen‡

(*College of Math. and Computer Science, Fuzhou University, Fuzhou 350116, Fujian, PR China*)

Abstract

A nonautonomous modified Leslie-Gower predator-prey model with non-monotonic functional response and a prey refuge is proposed and studied in this paper. Sufficient conditions which guarantee the permanence, extinction of the prey species and the global stability of the system are obtained, respectively. Also, by constructing a suitable Lyapunov function, some sufficient conditions are obtained for the existence of a unique globally attractive positive almost periodic solution of this model. Our results indicate that the prey refuge has positive effect on the coexistence of the species. Examples together with their numeric simulation show the feasibility of our main results.

Keywords predator; prey; permanence; global stability

2000 Mathematics Subject Classification 34D23; 92D25; 34D20; 34D40

1 Introduction

The traditional Leslie-Gower predator-prey model, which was proposed by Leslie ([1]), which takes the form:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = (r_1 - a_1 P - b_1 H)H, \quad \frac{\mathrm{d}P}{\mathrm{d}t} = \Big(r_2 - a_2 \frac{P}{H}\Big)P, \tag{1.1}$$

where H and P are the densities of prey species and the predator species at time t, respectively. A. Korobeinikov [2] showed the unique positive equilibrium of system (1.1) is globally stable.

*Manuscript received March 20, 2017

†The research was supported by the Natural Science Foundation of Fujian Province (2015J01012).

‡Corresponding author. E-mail: catharinella@foxmail.com

Lian and Xu [3] proposed the following delayed Leslie-Gower model with nonmonotonic functional response:

$$\begin{aligned} \dot{x} &= rx(t)\Big(1-\frac{x(t)}{k}\Big)-\frac{mx(t)}{a+x^2(t)}y(t),\\ \dot{y} &= y(t)\Big[s\Big(1-h\frac{y(t-\tau)}{x(t-\tau)+k_2}\Big)\Big]. \end{aligned} \tag{1.2}$$

By choosing the delay τ as a bifurcation parameter, the authors investigated the local asymptotic stability of the positive equilibrium and existence of local Hopf bifurcations. The supercritical stable Hopf bifurcations were also found by normal form theory.

With the aim of finding how the hunting delay affects the dynamics of the Leslie-Gower predator-prey model with nonmonotonic functional response, Jiao and Song [4] proposed the following model:

$$\begin{aligned} \dot{x} &= rx(t)\Big(1-\frac{x(t)}{k}\Big)-\frac{mx(t)}{a+x^2(t)}y(t-\tau),\\ \dot{y} &= s\Big(1-\frac{y(t)}{nx(t)}\Big)y(t). \end{aligned} \tag{1.3}$$

Such topics as the existence and local stability, global stability property of the positive equilibrium of the system were investigated. The authors also investigated the Hopf bifurcations of the system. In [5], the authors further investigated the dynamic behaviors of system (1.2) near the Bogdanov-Takens bifurcation point. By analyzing the characteristic equation associated with the non-hyperbolic equilibrium, the critical value of the delay inducing the Bogdanov-Takens bifurcation was obtained. They showed that the change of delay can result in heteroclinic orbit, homoclinic orbit and unstable limit cycle.

Yin et al [6] considered the spatial heterogeneity of the predators and preys distributions, and proposed a diffusive Leslie-Gower predator-prey system with nonmonotonic functional response:

$$\begin{aligned} \frac{\partial u}{\partial t} &= d_1\Delta u+u\Big(1-u-\frac{v}{au^2+bu+1}\Big), \quad x\in\Omega,\ t>0,\\ \frac{\partial v}{\partial t} &= d_2\Delta v+\eta v\Big(1-\frac{rv}{u}\Big), \quad x\in\Omega,\ t>0. \end{aligned} \tag{1.4}$$

The authors investigated the persistence of the model, the local and global stability of positive constant equilibrium and the Turing instability of the equilibrium.

On the other hand, more and more scholars have paid attention to the dynamic behaviors of the predator-prey system incorporating a prey refuge (see [7-10]). Chen et al [7] extended model (1.1) by incorporating a refuge protecting mH of the prey, where $m \in [0,1)$ is a constant, and proposed the following system:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = (r_1 - b_1 H)H - a_1(1-m)HP, \quad \frac{\mathrm{d}P}{\mathrm{d}t} = \Big(r_2 - a_2 \frac{P}{(1-m)H}\Big)P, \tag{1.5}$$

where $m \in [0,1)$ and $r_i, a_i,\ i=1,2,\ b_1$ are all positive constants. By constructing some suitable Lyapunov function, they showed that the unique positive equilibrium of the system is globally stable, consequently, prey refuge has no influence on the persistent property of the system.

Recently, Yue [8] proposed and studied the following modified Leslie-Gower predator-prey model with Holling-type II schemes and a prey refuge:

$$\begin{aligned} \dot{x}(t) &= x\Big(r_1 - b_1 x - \frac{a_1(1-m)y}{(1-m)x + k_1}\Big), \\ \dot{y}(t) &= y\Big(r_2 - \frac{a_2 y}{(1-m)x + k_2}\Big). \end{aligned} \tag{1.6}$$

The author obtained a set of sufficient conditions which ensure the global attractivity of a positive equilibrium.

It brings to our attention that to this day, still no scholar propose the modified Leslie-Gower predator prey model with nonmonotonic functional response and a prey refuge. This motivates us to propose the following system:

$$\begin{aligned} \dot{x}(t) &= x(t)\big(r_1(t) - b_1(t)x(t)\big) - \frac{c_1(t)(1-m(t))x(t)y(t)}{(1-m(t))^2(x(t))^2 + k_1(t)}, \\ \dot{y}(t) &= y(t)\Big(r_2(t) - \frac{c_2(t)y(t)}{(1-m(t))x(t) + k_2(t)}\Big), \end{aligned} \tag{1.7}$$

where $x(t)$ and $y(t)$ denote the densities of the prey and predator species at time t, respectively; in this system, we incorporate a refuge protecting $m(t)x$ of the prey, where $m(t) \in [0,1)$, this leaves $(1-m(t))x$ of the prey available to the predator.

Throughout this paper, we assume that:

(H_1) $k_i(t), c_i(t), r_i(t),\ i=1,2,\ m(t), b_1(t)$ are all continuous and strictly positive almost periodic functions defined on $[0,+\infty)$.

We consider (1.7) together with the following initial conditions

$$x(0) > 0, \quad y(0) > 0. \tag{1.8}$$

We mention here that the assumption of almost periodicity of the coefficients in (1.7) is a way of incorporating the time dependent variability of the environment, especially when the various components of the environment are periodic with not necessary commensurate periods (e.g. seasonal effects of weather, food supplies, mating habits, harvesting etc.). During the last decade, many scholars have investigated the almost periodic solution of the population dynamics, and some excellent results have been obtained, see [11-13] and the references therein.

The aim of this paper is to obtain a set of sufficient conditions which ensure the existence of a unique globally attractive positive almost periodic solution of system (1.7) with (1.8).

The rest of the paper is arranged as follows: In Section 2, we obtain sufficient conditions which guarantee the permanence of system (1.7). In Section 3, we obtain sufficient conditions which ensure the existence of a unique global attractive almost periodic solution of system (1.7). In Section 4, an example together with its numeric simulations illustrate the feasibility of the main results. We end this paper by a briefly discussion.

2 Permanence

For the rest of the paper, for a bounded continuous function g defined on R, let g^L and g^M be defined as

$$g^L = \inf_{t\in[0,+\infty)} g(t), \quad g^M = \sup_{t\in[0,+\infty)} g(t).$$

Lemma 2.1[14] *If $a>0,\ b>0$ and $\dot{x}\ge x(b-ax)$, when $t\ge 0$ and $x(0)>0$, we have*

$$\liminf_{t\to+\infty} x(t) \ge \frac{b}{a}.$$

If $a>0,\ b>0$ and $\dot{x}\le x(b-ax)$, when $t\ge 0$ and $x(0)>0$, we have

$$\limsup_{t\to+\infty} x(t) \le \frac{b}{a}.$$

Lemma 2.2 *The domain $R_+^2=\{(x,y)|\, x>0, y>0\}$ is invariant with respect to* (1.7).

Proof Since

$$x(t)=x(0)\exp\Big\{\int_0^t \Big(r_1(s)-b_1(s)x(s)-\frac{c_1(t)(1-m(s))y(s)}{(1-m(s))^2(x(s))^2+k_1(s)}\Big)\mathrm{d}s\Big\}>0,$$

$$y(t)=y(0)\exp\Big\{\int_0^t \Big(r_2(s)-\frac{c_2(s)y(s)}{(1-m(s))x(s)+k_2(s)}\Big)\mathrm{d}s\Big\}>0.$$

The assertion of the lemma follows immediately for all $t\in[0,+\infty)$.

Lemma 2.3 *Let $(x(t),y(t))^{\mathrm{T}}$ be any solution of system* (1.7) *with* (1.8), *then*

$$\begin{aligned}\limsup_{t\to+\infty} x(t) &\le \frac{r_1^M}{b_1^L} \stackrel{\text{def}}{=} M_1,\\ \limsup_{t\to+\infty} y(t) &\le \frac{r_2^M\big((1-m^L)M_1+k_2^M\big)}{c_2^L} \stackrel{\text{def}}{=} M_2.\end{aligned} \tag{2.1}$$

Proof Let $(x(t),y(t))^{\rm T}$ be any solution of system (1.7) with (1.8). From the first equation of system (1.7), it follows that

$$\dot{x}(t)\le x(r_1^M-b_1^Lx). \tag{2.2}$$

Applying Lemma 2.1 to (2.2), it immediately follows that

$$\limsup_{t\to+\infty}x(t)\le\frac{r_1^M}{b_1^L}\overset{\rm def}{=}M_1. \tag{2.3}$$

For any positive constant $\varepsilon>0$ small enough, it follows from (2.2) that there exists a $T_1>0$ such that

$$x(t)<M_1+\varepsilon\quad\text{for all } t\ge T_1. \tag{2.4}$$

For $t>T_1$, (2.4) together with the second equation of system (1.7) leads to

$$\dot{y}(t)\le y\Big(r_2^M-\frac{c_2^Ly}{(1-m^L)(M_1+\varepsilon)+k_2^M}\Big). \tag{2.5}$$

Applying Lemma 2.1 to (2.5), it immediately follows that

$$\limsup_{t\to+\infty}y(t)\le\frac{r_2^M\big((1-m^L)(M_1+\varepsilon)+k_2^M\big)}{c_2^L}.$$

Set $\varepsilon\to0$, then

$$\limsup_{t\to+\infty}y(t)\le\frac{r_2^M\big((1-m^L)M_1+k_2^M\big)}{c_2^L}\overset{\rm def}{=}M_2. \tag{2.6}$$

This ends the proof of Lemma 2.3.

Lemma 2.4 *Let* $(x(t),y(t))^{\rm T}$ *be any solution of system* (1.7) *with* (1.8). *Assume that*

(H_2) $r_1^Lk_1^L>c_1^M(1-m^L)M_2$

holds, then

$$\begin{aligned}\liminf_{t\to+\infty}x(t)&\ge\frac{r_1^Lk_1^L-c_1^M(1-m^L)M_2}{k_1^Lb^M}\overset{\rm def}{=}m_1,\\ \liminf_{t\to+\infty}y(t)&\ge\frac{r_2^Lk_2^L}{c_2^M}\overset{\rm def}{=}m_2.\end{aligned} \tag{2.7}$$

Proof For ε small enough, it follows from (2.1) that there exists a $T_2>T_1$ such that

$$y(t)<M_2+\varepsilon\quad\text{for all } t\ge T_2. \tag{2.8}$$

Let $(x(t),y(t))^{\rm T}$ be any solution of system (1.7) with (1.8). For $t>T_2$, (2.8) together with the first equation of system (1.7) leads to

$$\dot{x}(t) \geq x\Big(r_1^L - \frac{c_1^M(1-m^L)(M_2+\varepsilon)}{k_1^L} - b_1^M x(t)\Big). \tag{2.9}$$

Applying Lemma 2.1 to (2.9), it immediately follows that

$$\liminf_{t\to+\infty} x(t) \geq \frac{r_1^L - \dfrac{c_1^M(1-m^L)(M_2+\varepsilon)}{k_1^L}}{b_1^M}. \tag{2.10}$$

Set $\varepsilon \to 0$, then

$$\liminf_{t\to+\infty} x(t) \geq \frac{r_1^L k_1^L - c_1^M(1-m^L)M_2}{k_1^L b^M} \stackrel{\text{def}}{=} m_1. \tag{2.11}$$

From the second equation of system (1.7), we have

$$\dot{y}(t) \geq y(t)\Big(r_2^L - \frac{c_2^M y(t)}{k_2^L}\Big). \tag{2.12}$$

Apply Lemma 2.1 to (2.12), then

$$\liminf_{t\to+\infty} y(t) \geq \frac{r_2^L k_2^L}{c_2^M} \stackrel{\text{def}}{=} m_2. \tag{2.13}$$

As a direct corollary of Lemmas 2.3 and 2.4, we have:

Theorem 2.1 *Under assumptions* (H_1) and (H_2), *system* (1.7) *with* (1.8) *is permanent.*

Remark 2.1 Noting that if $m(t)$ is enough large, then condition (H_2) always holds, and from Theorem 2.1, system (1.7) is permanent, which means that prey refuge could improve the change of the survival of prey species, and finally leads to the coexistence of the two species.

Define

$$\Gamma^\varepsilon = \{(x,y)^{\mathrm{T}} \in R^2 |\, m_1 - \varepsilon < x < M_1 + \varepsilon, m_2 - \varepsilon < y < M_2 + \varepsilon\},$$

where $\varepsilon > 0$ is sufficiently small. By using (2.2), (2.5), (2.9) and (2.12), similarly to the analysis of Lemma 2.3 in [11], we have:

Theorem 2.2 *Under assumptions* (H_1) *and* (H_2), *the set* Γ^ε *is invariant with respect to* (1.7).

3 Extinction of Prey Species

The aim of this section is to obtain a set of sufficient conditions which ensure the extinction of the prey species, more precisely, we have the following result.

Theorem 3.1 *Assume that*

(H_3)

$$r_1^M < \frac{c_1^L(1-m^M)m_2}{M_1^2+k_1^M} \tag{3.1}$$

holds. Then the prey species will be driven to extinction.

Remark 3.1 Condition (H_3) shows that despite the refuge protect part of the prey species, if the net birth rate of the prey species is lower enough, then the prey species still be driven to extinction.

Proof of Theorem 3.1 Condition (H_3) implies that one could choose an $\varepsilon > 0$ ($\varepsilon < \frac{1}{2}m_2$) enough small such that

$$r_1^M < \frac{c_1^L(1-m^M)(m_2-\varepsilon)}{(M_1+\varepsilon)^2+k_1^M} \tag{3.2}$$

holds. For this ε, from the proof of Theorem 2.1, there exists an enough large $T_3 > T_2$ such that for all $t > T_3$

$$x(t) < M_1+\varepsilon, \quad y(t) > m_2 - \frac{1}{2}\varepsilon. \tag{3.3}$$

Then, from the first equation of system (1.7), one has

$$\begin{aligned}\dot{x}(t) &= x(t)\big(r_1(t)-b_1(t)x(t)\big) - \frac{c_1(t)(1-m(t))x(t)y(t)}{(1-m(t))^2(x(t))^2+k_1(t)} \\ &\le x(t)\Big(r_1^M - \frac{c_1^L(1-m^M)(m_2-\varepsilon)}{(1-m^L)^2(M_1+\varepsilon)^2+k_1^M}\Big).\end{aligned} \tag{3.4}$$

Thus, it immediately follows from (3.2) that

$$x(t) \le x(T_3)\exp\Big\{\int_{T_3}^t \Big(r_1^M - \frac{c_1^L(1-m^M)(m_2-\varepsilon)}{(1-m^L)^2(M_1+\varepsilon)^2+k_1^M}\Big)\mathrm{d}t\Big\} \to 0 \quad \text{as } t\to+\infty.$$

This ends the proof of Theorem 3.1.

4 Global Attractivity

Before we state the main result of this section, we introduce some notations. Set

$$\begin{aligned}A_1(t) &\overset{\text{def}}{=} b_1(t) - \frac{2c_1(t)(1-m(t))M_2M_1}{\big((1-m(t))^2m_1^2+k_1(t)\big)^2} - \frac{c_2(t)(1-m(t))M_2}{\big((1-m(t))m_1+k_2(t)\big)^2}, \\ A_2(t) &\overset{\text{def}}{=} \frac{c_2(t)}{(1-m(t))M_1+k_2(t)} - \frac{c_1(t)(1-m(t))}{(1-m(t))^2m_1^2+k_1(t)}.\end{aligned} \tag{4.1}$$

Theorem 4.1 *In addition to* (H_1)-(H_2), *assume further that*

$$\liminf_{t\to+\infty}\big\{A_1(t), A_2(t)\big\} > 0, \tag{4.2}$$

then for any positive solutions $(x(t), y(t))^{\mathrm{T}}$ *and* $(x^*(t), y^*(t))^{\mathrm{T}}$ *of system* (1.7), *one has*

$$\lim_{t\to+\infty}(|x(t)-x^*(t)|+|y(t)-y^*(t)|)=0. \tag{4.3}$$

As a direct corollary of Theorem 4.1, we have:

Corollary 4.1 *In addition to* (H_1)-(H_2), *assume further that*

$$\begin{aligned} b_1^L &> \frac{2c_1^M(1-m^L)M_2M_1}{\left(k_1^L\right)^2}+\frac{c_2^M(1-m^L)M_2}{\left(k_2^L\right)^2}, \\ \frac{c_2^L}{M_1+k_2^M} &> \frac{c_1^M(1-m^L)}{k_1^L}, \end{aligned} \tag{4.4}$$

then for any positive solutions $(x(t), y(t))^{\mathrm{T}}$ *and* $(x^*(t), y^*(t))^{\mathrm{T}}$ *of system* (1.7), *one has*

$$\lim_{t\to+\infty}(|x(t)-x^*(t)|+|y(t)-y^*(t)|)=0.$$

Remark 4.1 Note that if $m(t)$ is enough large, then inequalities (4.4) always holds, consequently, the predator and prey species could be coexistence in a stable state. This means that prey refuge has positive effect on the stability property of the system.

Proof of Theorem 4.1 Condition (4.2) implies that there exists an enough small positive constant ε (without loss of generality, we may assume that $\varepsilon < \frac{1}{2}\{m_1, m_2\}$) such that

$$\begin{aligned} A_1(\varepsilon,t) &= b_1(t)-\frac{2c_1(t)(1-m(t))(M_2+\varepsilon)(M_1+\varepsilon)}{\left((1-m(t))^2(m_1-\varepsilon)^2+k_1(t)\right)^2}-\frac{c_2(t)(1-m(t))(M_2+\varepsilon)}{\left((1-m(t))(m_1-\varepsilon)+k_2(t)\right)^2} \\ &\geq \varepsilon, \\ A_2(\varepsilon,t) &= \frac{c_2(t)}{(1-m(t))(M_1+\varepsilon)+k_2(t)}-\frac{c_1(t)(1-m(t))}{(1-m(t))^2(m_1-\varepsilon)^2+k_1(t)}\geq\varepsilon. \end{aligned} \tag{4.5}$$

For two arbitrary positive solutions $(x(t), y(t))^{\mathrm{T}}$ and $(x^*(t), y^*(t))^{\mathrm{T}}$ of system (1.7), for the above $\varepsilon > 0$, it follows from (2.1) and (2.7) that there exists an enough large T, such that for all $t \geq T$

$$\begin{aligned} &x(t), x^*(t) < M_1+\varepsilon, \;\; y(t), y^*(t) < M_2+\varepsilon, \\ &x(t), x^*(t) > m_1-\varepsilon, \;\; y(t), y^*(t) > m_2-\varepsilon. \end{aligned} \tag{4.6}$$

Now we let

$$V_1(t)=|\ln x(t)-\ln x^*(t)|, \quad V_2(t)=|\ln y(t)-\ln y^*(t)|. \tag{4.7}$$

Then, by using (4.6), for $t > T$, we have

$$\begin{aligned}
D^+V_1(t) \le\ & \operatorname{sgn}\big(x(t)-x^*(t)\big)\Big(-b_1(t)x(t)-\frac{c_1(t)(1-m(t))y(t)}{(1-m(t))^2(x(t))^2+k_1(t)}\\
&+b_1(t)x^*(t)+\frac{c_1(t)(1-m(t))y^*(t)}{(1-m(t))^2(x^*(t))^2+k_1(t)}\Big)\\
\le\ & -b_1(t)|x(t)-x^*(t)|+\operatorname{sgn}(x(t)-x^*(t))c_1(t)(1-m(t))\times\\
&\Big(-\frac{y(t)}{(1-m(t))^2(x(t))^2+k_1(t)}+\frac{y^*(t)}{(1-m(t))^2(x(t))^2+k_1(t)}\\
&-\frac{y^*(t)}{(1-m(t))^2(x(t))^2+k_1(t)}+\frac{y^*(t)}{(1-m(t))^2(x^*(t))^2+k_1(t)}\Big)\\
&+\frac{c_1(t)(1-m(t))y^*(t)(x^*(t)+x(t))}{\big((1-m(t))^2(x(t))^2+k_1(t)\big)\big((1-m(t))^2(x^*(t))^2+k_1(t)\big)}|x(t)-x^*(t)|\\
\le\ & -b_1(t)|x(t)-x^*(t)|+\frac{c_1(t)(1-m(t))}{(1-m(t))^2(m_1-\varepsilon)^2+k_1(t)}|y(t)-y^*(t)|\\
&+\frac{2c_1(t)(1-m(t))(M_2+\varepsilon)(M_1+\varepsilon)}{\big((1-m(t))^2(m_1-\varepsilon)^2+k_1(t)\big)^2}|x(t)-x^*(t)|.
\end{aligned}$$

Similarly,

$$\begin{aligned}
D^+V_2(t) =\ & \operatorname{sgn}\big(y(t)-y^*(t)\big)\Big(-\frac{c_2(t)y(t)}{(1-m(t))x(t)+k_2(t)}+\frac{c_2(t)y^*(t)}{(1-m(t))x^*(t)+k_2(t)}\Big)\\
\le\ & -\frac{c_2(t)}{(1-m(t))(M_1+\varepsilon)+k_2(t)}|y(t)-y^*(t)|\\
&+\frac{c_2(t)(1-m(t))(M_2+\varepsilon)}{\big((1-m(t))(m_1-\varepsilon)+k_2(t)\big)^2}|x(t)-x^*(t)|.
\end{aligned}$$

Now set

$$V(t)=V_1(t)+V_2(t).$$

Then

$$\begin{aligned}
D^+V(t) \le\ & -\Big(b_1(t)-\frac{2c_1(t)(1-m(t))(M_2+\varepsilon)(M_1+\varepsilon)}{\big((1-m(t))^2(m_1-\varepsilon)^2+k_1(t)\big)^2}\\
&-\frac{c_2(t)(1-m(t))(M_2+\varepsilon)}{\big((1-m(t))(m_1-\varepsilon)+k_2(t)\big)^2}\Big)|x(t)-x^*(t)|\\
&-\Big(\frac{c_2(t)}{(1-m(t))(M_1+\varepsilon)+k_2(t)}-\frac{c_1(t)(1-m(t))}{(1-m(t))^2(m_1-\varepsilon)^2+k_1(t)}\Big)|y(t)-y^*(t)|\\
=\ & -A_1(\varepsilon,t)|x(t)-x^*(t)|-A_2(\varepsilon,t)|y(t)-y^*(t)|. \qquad (4.8)
\end{aligned}$$

Integrating both sides of (4.8) on interval $[T,t)$,

$$V(t)-V(T)\le\int_T^t\big[-A_1(\varepsilon,s)|x(s)-x_1(s)|-A_2(\varepsilon,s)|y(s)-y_1(s)|\big]\mathrm{d}s\quad \text{for } t\ge T. \qquad (4.9)$$

It follows from (4.5) that

$$V(t)+\varepsilon\int_T^t\big[|x(s)-x_1(s)|+|y(s)-y_1(s)|\big]\mathrm{d}s\le V(T)\quad\text{for } t\ge T.$$

Therefore, $V(t)$ is bounded on $[T,+\infty)$ and also

$$\int_T^t\big[|x(s)-x_1(s)|+|y(s)-y_1(s)|\big]\mathrm{d}s<+\infty. \tag{4.10}$$

By Theorem 2.1, $|x(t)-x^*(t)|,|y(t)-y^*(t)|$ are bounded on $[T,+\infty)$. On the other hand, it is easy to see that $\dot{x}(t)$, $\dot{y}(t)$, $\dot{x}_1(t)$ and $\dot{y}_1(t)$ are bounded for $t\ge T$. Therefore, $|x(t)-x^*(t)|,|y(t)-y^*(t)|$ are uniformly continuous on $[T,+\infty)$. By Barbălat lemma, one can conclude that

$$\lim_{t\to+\infty}\big[|x(t)-x^*(t)|+|y(t)-y^*(t)|\big]=0.$$

This ends the proof of the Theorem 4.1.

5 Almost Periodic Solution

This section deals with the almost periodic solution of system (1.7).

Let

$$x(t)=\mathrm{e}^{\widetilde{x}(t)},\quad y(t)=\mathrm{e}^{\widetilde{y}(t)}.$$

then system (1.7) becomes

$$\begin{aligned}\dot{\widetilde{x}}(t)&=r_1(t)-b_1(t)\mathrm{e}^{\widetilde{x}(t)}-\frac{c_1(t)(1-m(t))\mathrm{e}^{\widetilde{y}(t)}}{(1-m(t))^2\mathrm{e}^{2\widetilde{x}(t)}+k_1(t)},\\ \dot{\widetilde{y}}(t)&=r_2(t)-\frac{c_2(t)\mathrm{e}^{\widetilde{y}(t)}}{(1-m(t))\mathrm{e}^{\widetilde{x}(t)}+k_2(t)}.\end{aligned} \tag{5.1}$$

By the relationship of systems (5.1) and (1.7), corresponding to Theorem 2.2, one could easily obtain the following results.

Lemma 5.1 *Under assumptions* (H_1) *and* (H_2)*, the set* Γ_1^ε *defined by*

$$\Gamma_1^\varepsilon=\big\{F=(x,y)\in R^2|\ \ln(m_1-\varepsilon)\le\widetilde{x}\le\ln(M_1+\varepsilon),\ln(m_2-\varepsilon)\le\widetilde{y}\le\ln(M_2+\varepsilon)\big\}$$

is invariant with respect to (5.1).

It is obvious that if system (5.1) exists an almost periodic solution, then system (1.7) also exists a positive almost periodic solution.

Now we are in the position of stating the main results of this section:

Theorem 5.1 *In addition to assumptions* (H_1) *and* (H_2)*, assume further that there exists a* $\mu > 0$ *such that*

$$\min\{A_1(t), A_2(t)\} \geq \mu, \tag{5.2}$$

where $A_i(t)$*,* $i = 1, 2$ *are defined by* (4.1)*, then system* (1.7) *admits a unique globally attractive strictly positive* (*componentwise*) *almost periodic solution* $(x(t), y(t))$*,* $t \in R$*.*

Proof For $\varepsilon > 0$ enough small, from (5.2) one could obtain that

$$\min\{A_1(\varepsilon, t), A_2(\varepsilon, t)\} \geq \frac{\mu}{2}, \tag{5.3}$$

where $A_i(\varepsilon, t),\ i = 1, 2$ are defined by (4.5) and (4.6).

For $(X, Y) \in R_+^2$, we define $\|(X, Y)\| = X + Y$.

We first shows that system (5.1) has a unique almost periodic solution that is uniformly asymptotically stable in Γ_1^ε.

Consider the product system of system (5.1)

$$\begin{cases} \dot{\widetilde{x}}(t) = r_1(t) - b_1(t)\mathrm{e}^{\widetilde{x}(t)} - \dfrac{c_1(t)(1 - m(t))\mathrm{e}^{\widetilde{y}(t)}}{(1 - m(t))^2\mathrm{e}^{2\widetilde{x}(t)} + k_1(t)}, \\ \dot{\widetilde{y}}(t) = r_2(t) - \dfrac{c_2(t)\mathrm{e}^{\widetilde{y}(t)}}{(1 - m(t))\mathrm{e}^{\widetilde{x}(t)} + k_2(t)}, \\ \dot{\widetilde{u}}(t) = r_1(t) - b_1(t)\mathrm{e}^{\widetilde{u}(t)} - \dfrac{c_1(t)(1 - m(t))\mathrm{e}^{\widetilde{v}(t)}}{(1 - m(t))^2\mathrm{e}^{2\widetilde{u}(t)} + k_1(t)}, \\ \dot{\widetilde{v}}(t) = r_2(t) - \dfrac{c_2(t)\mathrm{e}^{\widetilde{v}(t)}}{(1 - m(t))\mathrm{e}^{\widetilde{u}(t)} + k_2(t)}. \end{cases} \tag{5.4}$$

Suppose $\widetilde{Z} = (\widetilde{x}, \widetilde{y})$, $\widetilde{W} = (\widetilde{u}, \widetilde{v})$ are any two solutions of system (5.1) defined on $[0, +\infty) \times \Gamma_1^\varepsilon \times \Gamma_1^\varepsilon$.

Consider a Lyapunov function defined on $[0, +\infty) \times \Gamma_1^\varepsilon \times \Gamma_1^\varepsilon$ as follows

$$V(t, \widetilde{Z}, \widetilde{W}) = |\widetilde{u}(t) - \widetilde{x}(t)| + |\widetilde{v}(t) - \widetilde{y}(t)|.$$

It then follows

$$A\|\widetilde{Z} - \widetilde{W}\| \leq V(t, \widetilde{Z}, \widetilde{W}) \leq B\|\widetilde{Z} - \widetilde{W}\|,$$

where $A = B = 1$.

In addition

$$\begin{aligned} &|V(t, \widetilde{Z}_1, \widetilde{W}_1) - V(t, \widetilde{Z}_2, \widetilde{W}_2)| \\ \leq\ &|\widetilde{x}_1(t) - \widetilde{x}_2(t)| + |\widetilde{y}_1(t) - \widetilde{y}_2(t)| + |\widetilde{u}_1(t) - \widetilde{u}_2(t)| + |\widetilde{v}_1(t) - \widetilde{v}_2(t)| \\ =\ &\|\widetilde{Z}_1 - \widetilde{Z}_2\| + \|\widetilde{W}_1 - \widetilde{W}_2\|. \end{aligned}$$

Finally, calculate the right derivative $D^+V(t)$ of $V(t)$ along the solutions of system (5.1), by using Lemma 3.2, similarly to the analysis of Theorem 2.1, by using (5.3), we can obtain:

$$\begin{aligned} D^+V(t) &\le -A_1(\varepsilon,t)|\mathrm{e}^{\widetilde{u}(t)}-\mathrm{e}^{\widetilde{x}(t)}| - A_2(\varepsilon,t)|\mathrm{e}^{\widetilde{v}(t)}-\mathrm{e}^{\widetilde{y}(t)}| \\ &\le -\frac{\mu}{2}\big(|\mathrm{e}^{\widetilde{u}(t)}-\mathrm{e}^{\widetilde{x}(t)}| + |\mathrm{e}^{\widetilde{v}(t)}-\mathrm{e}^{\widetilde{y}(t)}|\big). \end{aligned}$$

Note that

$$\mathrm{e}^{\widetilde{u}(t)}-\mathrm{e}^{\widetilde{x}(t)} = \mathrm{e}^{\widetilde{\xi}(t)}\big(\widetilde{u}(t)-\widetilde{x}(t)\big), \quad \mathrm{e}^{\widetilde{v}(t)}-\mathrm{e}^{\widetilde{y}(t)} = \mathrm{e}^{\widetilde{\varpi}(t)}\big(\widetilde{v}(t)-\widetilde{y}(t)\big).$$

Here, $\widetilde{\xi}(t)$ is a bounded function between $\widetilde{u}(t)$ and $\widetilde{x}(t)$ while $\widetilde{\varpi}(t)$ is a bounded function between $\widetilde{v}(t)$ and $\widetilde{y}(t)$.

Then we have

$$\begin{aligned} D^+V(t) &\le -\frac{\mu}{2}\big[(m_1-\varepsilon)|\widetilde{u}(t)-\widetilde{x}(t)| + (m_2-\varepsilon)|\widetilde{v}(t)-\widetilde{y}(t)|\big] \\ &\le -\frac{m\mu}{2}\big\{|\widetilde{u}(t)-\widetilde{x}(t)| + |\widetilde{v}(t)-\widetilde{y}(t)|\big\} \\ &\le -\frac{m\mu}{2}V(t,\widetilde{Z},\widetilde{W}), \end{aligned}$$

where $m = \min\{m_1-\varepsilon, m_2-\varepsilon\}$. Hence, the condition (3) of Lemma 5.1 is also satisfied.

The above analysis shows that all the conditions of Theorem 6.3 in [15] hold. Thus, system (5.1) has a unique almost periodic solution $(\widetilde{x}^*,\widetilde{y}^*)$ which is uniformly asymptotically stable in Γ_1^ε. Hence, system (1.7) has a unique positive almost periodic solution $(\mathrm{e}^{\widetilde{x}^*},\mathrm{e}^{\widetilde{y}^*})$ which is uniformly asymptotically stable in Γ^ε. The global attractivity of the positive almost periodic solution of system (1.7) is a direct corollary of Theorem 4.1. The proof is completed.

6 Numeric Examples

Now let's consider the following examples.

Example 6.1

$$\begin{aligned} \dot{x}(t) &= x\Big(8+\sin(\sqrt{3}t) - 3x - \frac{(1-0.9)(4+\cos(\sqrt{2}t))xy}{(1-0.9)^2x^2+5}\Big), \\ \dot{y}(t) &= y\Big(1-\frac{2y}{(1-0.9)x+3}\Big). \end{aligned} \tag{6.1}$$

One could easily verify the system satisfies the conditions of Corollary 4.1 and Theorem 5.1, then system (6.1) admits a unique globally attractive strictly positive (componentwise) almost periodic solution $(x^*(t),y^*(t))$, $t\in R$. Numeric simulations (Figures 6.1, 6.2) also support this finds.

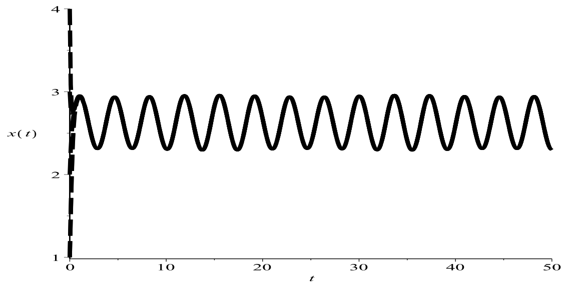

Figure 1: Dynamic behavior of the first component $x(t)$ of the solution $(x(t), y(t))$ of system (6.1) with the initial conditions $(x(0), y(0)) = (1, 1)$, $(0.3, 2)$, $(0.2, 2)$, $(0.1, 3)$ and $(0.2, 0.2)$, respectively.

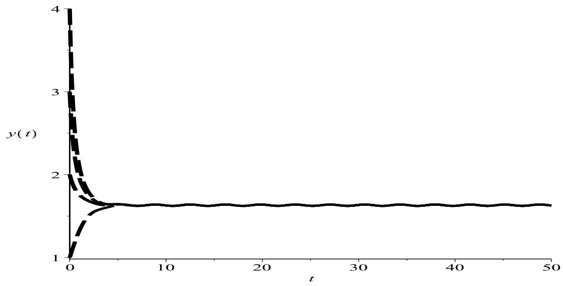

Figure 2: Dynamic behavior of the second component $y(t)$ of the solution $(x(t), y(t))$ of system (6.1) with the initial conditions $(x(0), y(0)) = (1, 1)$, $(0.3, 2)$, $(0.2, 2)$, $(0.1, 3)$ and $(0.2, 0.2)$, respectively.

Example 6.2

$$\begin{aligned} \dot{x}(t) &= x(t)\big(2 + \sin(\sqrt{3}t) - 6x(t)\big) - \frac{0.9(4 + \cos(\sqrt{2}t))y(t)x(t)}{0.81(x(t))^2 + 3}, \\ \dot{y}(t) &= y\Big(4 - \frac{10y}{(1 - 0.1)x + 10}\Big). \end{aligned} \tag{6.2}$$

One could easily verify the system satisfies the conditions of Theorem 3.1, hence the prey species will be driven to extinction. Numeric simulations (Figure 6.3) also support this finds.

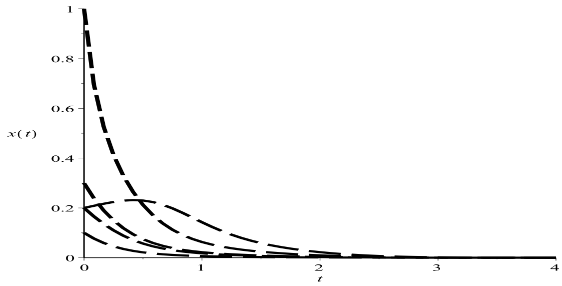

Figure 3: Dynamic behavior of the first component $x(t)$ of the solution $(x(t), y(t))$ of system (6.2) with the initial conditions $(x(0), y(0)) = (1, 1)$, $(0.3, 2)$, $(0.2, 2)$, $(0.1, 3)$ and $(0.2, 0.2)$, respectively.

7 Conclusion

In this paper, stimulated by the works of Lian and Xu [3], Chen et al [7] and Yue [8], we propose a nonautonomous modified Leslie-Gower predator-prey system with nonmonotonic functional response and a prey refuge. We pay attention to the persistent and stability property of the system, as well as that of the existence of a unique globally attractive almost periodic solution.

One interesting finding is that for system (1.7), the prey refuge plays an important role on the persistent property of the system, One could always provide enough large prey refuge, such that two species could be coexist in a stable state.

We mention here that delay is one of the most important factor which could lead to the bifurcation of the system, and it seems interesting to study the almost periodic solution of the delayed nonautonomous modified Leslie-Gower predator-prey system with non-monotonic functional response and a prey refuge. We leave this for future investigation.

References

[1] P.H. Leslie, A stochastic model for studying the properties of certain biological systems by numerical methods, *Biometrika*, **45**:1-2(1958),16-31.

[2] A. Korobeinikov, A Lyapunov function for Leslie-Gower predator-prey models, *Applied Mathematics Letters*, **14**:6(2001),697-699.

[3] F. Lian and Y. Xu, Hopf bifurcation analysis of a predator-prey system with Holling type IV functional response and time delay, *Applied Mathematics and Computation*, **215**:4(2009),1484-1495.

[4] J. Jiang, Y. Song, Stability and bifurcation analysis of a delayed Leslie-Gower predator-prey system with nonmonotonic functional response, *Abstract Appl. Anal.*, 2013;2013:19 pages. Article ID 152459.

[5] J. Jiang, Y.L. Song, Delay induced Bogdanov-Takens bifurcation in a Leslie Gower predator prey model with nonmonotonic functional response, *Communications in Nonlinear Science and Numerical Simulation*, **19**:7(2014),2454-2465.

[6] H.W. Yin, J.X. Zhou, X.Y. Xiao, et al, Analysis of a diffusive Leslie-Gower predator-prey model with nonmonotonic functional response, *Chaos, Solitons* & *Fractals*, **65**(2014),51-61.

[7] F.D. Chen, L.J. Chen and X.D. Xie, On a Leslie-Gower predator-prey model incorporating a prey refuge, *Nonlinear Analysis: Real World Applications*, **10**:5(2009),2905-2908.

[8] Q. Yue, Dynamics of a modified Leslie-Gower predator-prey model with Holling-type II schemes and a prey refuge, *SpringerPlus*, **5**:1(2016),1-12.

[9] Y. Wu, F. Chen, W. Chen & Y. Lin, Dynamic behaviors of a nonautonomous discrete predator-prey system incorporating a prey refuge and Holling type II functional response, *Discrete Dynamics in Nature* & *Society*, **2012**(2012),1951-1965.

[10] L.J. Chen, F.D. Chen and L.J. Chen, Qualitative analysis of a predator-prey model with Holling type II functional response incorporating a constant prey refuge, *Nonlinear Analysis: Real World Applications*, **11**:1(2010),246-252.

[11] F.D. Chen, C.L. Shi, Global attractivity in an almost periodic multi-species nonlinear ecological model, *Applied Mathematics and Computation*, **180**:1(2006),376-392.

[12] Z. Li, M.A. Chen, F.D. Chen, Almost periodic solutions of a discrete almost periodic logistic equation with delay, *Applied Mathematics and Computation*, **232**:1(2014),743-751.

[13] X. Xie, C.H. Zhang, X.X. Chen, J.Y. Chen, Almost periodic sequence solution of a discrete Hassell-Varley predator-prey system with feedback control, *Applied Mathematics and Computation*, **268**:1(2015),35-51.

[14] F. Chen, Z. Li, Y.J. Huang, Note on the permanence of a competitive system with infinite delay and feedback controls, *Nonlinear Analysis: Real World Applications*, **8**(2007),680-687.

[15] C.Y. He, Almost Periodic Differential Equations, Beijing: Higher Education Press, 1992. (in Chinese)

(*edited by Mengxin He*)