Ann. of Appl. Math.
34:1(2018), 47-57

POSITIVE PERIODIC SOLUTIONS OF THE FIRST-ORDER SINGULAR DISCRETE SYSTEMS[*†]

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Abstract

In this paper, we study two kinds of first-order singular discrete systems. By the fixed point index theory, we investigate the existence and multiplicity of positive periodic solutions of the systems.

Keywords positive periodic solutions; singular discrete systems; delay; fixed point index

2000 Mathematics Subject Classification 34B15; 34C25

1 Introduction

Let $T > 3$ be an integer. In this paper, we are concerned with the existence and multiplicity of positive T-periodic solutions of the following singular discrete systems

$$\Delta u_i(t) = -a_i(t)g_i(\mathbf{u}(t))u_i(t) + \lambda b_i(t)f_i(\mathbf{u}(t-\tau(t))), \quad t \in \mathbb{Z},\ i = 1, 2, \cdots, n \tag{1.1}$$

and

$$\Delta u_i(t) = a_i(t)g_i(\mathbf{u}(t))u_i(t) - \lambda b_i(t)f_i(\mathbf{u}(t-\tau(t))), \quad t \in \mathbb{Z},\ i = 1, 2, \cdots, n, \tag{1.2}$$

where $\mathbf{u} = (u_1, \cdots, u_n) \in \mathbb{R}^n$, $a_i, b_i : \mathbb{Z} \to [0, \infty)$ are T-periodic functions with

$$\sum_{t=0}^{T-1} a_i(t) > 0, \quad \sum_{t=0}^{T-1} b_i(t) > 0;$$

$g_i \in C(\mathbb{R}^n_+, [0, \infty))$ and $f_i : \mathbb{R}^n_+ \backslash \{\mathbf{0}\} \to [0, \infty)$ are continuous, $i = 1, 2, \cdots, n$; $\tau : \mathbb{Z} \to \mathbb{Z}$ is a T-periodic function and λ is a positive parameter.

In the past few years, there has been considerable interest in the existence of periodic solutions of equations

$$u'(t) = a(t)g(u(t))u(t) - \lambda b(t)f(u(t-\tau(t))) \tag{1.3}$$

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*Manuscript received July 2, 2017

†Supported by the National Natural Science Foundation of China (Grant No.11601011).

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and

$$u'(t) = -a(t)g(u(t))u(t) + \lambda b(t)f(u(t-\tau(t))), \tag{1.4}$$

where $a, b \in C(\mathbb{R}, [0,\infty))$ are T-periodic functions with

$$\int_0^T a(t)\mathrm{d}t > 0, \quad \int_0^T b(t)\mathrm{d}t > 0,$$

and τ is a continuous T-periodic function. Equations (1.3) and (1.4) have been proposed as models for a variety of physiological processes and conditions including production of blood cells, respiration, and cardiac arrhythmias. See for example, [1-8,12] and the references therein. On the other hand, many authors paid their attention to the existence of positive periodic solutions of singular systems of the first-order and second-order differential equations, see Chu [9], Jiang [10], Wang [11,12] and the references therein. It has been shown that many results of nonsingular systems still valid for singular cases.

Let

$$\mathbb{R}_+ = [0,\infty), \quad \mathbb{R}_+^n = \prod_{i=1}^n \mathbb{R}_+,$$

and for any $\mathbf{u} = (u_1, \cdots, u_n) \in \mathbb{R}_+^n$,

$$\|\mathbf{u}\| = \sum_{i=1}^n |u_i|.$$

Recently, Wang [12] studied the existence and multiplicity of positive periodic solutions of the following singular non-autonomous n-dimensional system

$$x_i'(t) = -a_i(t)x_i(t) + \lambda b_i(t)f_i(x_1(t), \cdots, x_n(t)), \quad i = 1, \cdots, n \tag{1.5}$$

under assumptions

(H1) $a_i, b_i \in C(\mathbb{R}, [0,\infty))$ are ω-periodic functions such that $\int_0^\omega a_i(t)\mathrm{d}t > 0$, $\int_0^\omega b_i(t)\mathrm{d}t > 0$, $i = 1, \cdots, n$;

(H2) $f_i : \mathbb{R}_+^n \backslash \{\mathbf{0}\} \to (0,\infty)$ are continuous, $i = 1, \cdots, n$.

By using Krasnoselskii fixed point theorem in a cone, the author established the existence and multiplicity of positive periodic solutions of (1.5) with superlinearity or sublinearity assumptions at infinity for an appropriately chosen parameter.

However, to the best of our knowledge, the existence results of positive periodic solutions for first-order discrete systems (1.1) and (1.2) with singular nonlinearities are relatively little. Motivated by the above considerations, in this paper, we study the existence and multiplicity of positive T-periodic solutions of singular discrete systems (1.1) and (1.2). Obviously, (1.1) is a discrete analogue of system (1.5) when $g_i \equiv 1$, $i = 1, 2, \cdots, n$ and $\tau \equiv 0$, and we are interested in establishing the similar results as [12, Theorem 1.1] for systems (1.1) and (1.2).

We make the following assumptions:

(C1) $a_i, b_i : \mathbb{Z} \to [0, \infty)$ are T-periodic functions with $\sum\limits_{t=0}^{T-1} a_i(t) > 0$, $\sum\limits_{t=0}^{T-1} b_i(t) > 0$, $i = 1, 2, \cdots, n$; $\tau : \mathbb{Z} \to \mathbb{Z}$ is a T-periodic function.

(C2) $g_i \in C(\mathbb{R}_+^n, [0, \infty))$ satisfies $0 < l_i \le g_i(\mathbf{u}) \le L_i < \infty$, $f_i : \mathbb{R}_+^n \backslash \{\mathbf{0}\} \to (0, \infty)$ is continuous, $i = 1, 2, \cdots, n$.

(C3) $0 \le l_i a_i(t) \le L_i a_i(t) < 1,\ t \in \mathbb{T} := \{0, 1, \cdots, T-1\},\ i = 1, 2, \cdots, n.$

Our main results can be stated as below.

Theorem 1.1 *Let* (C1)-(C3) *hold. Suppose* $\lim\limits_{\|\mathbf{u}\| \to 0} f_i(\mathbf{u}) = \infty$ *for some* $i = 1, 2, \cdots, n$, *then:*

(i) *If* $\lim\limits_{\|\mathbf{u}\| \to \infty} \frac{f_i(\mathbf{u})}{\|\mathbf{u}\|} = 0$, $i = 1, 2, \cdots, n$, *then for all* $\lambda > 0$, (1.1) *admits a positive periodic solution.*

(ii) *If* $\lim\limits_{\|\mathbf{u}\| \to \infty} \frac{f_i(\mathbf{u})}{\|\mathbf{u}\|} = \infty$, $i = 1, 2, \cdots, n$, *then* (1.1) *admits two positive periodic solutions for* $\lambda > 0$ *sufficiently small.*

(iii) *There exists a* $\lambda_0 > 0$ *such that* (1.1) *admits a positive periodic solution for* $0 < \lambda < \lambda_0$.

Remark 1.1 Theorem 1.1, which improves the corresponding ones established for single difference equations in [17-21], is the discrete analogues of [12, Theorem 1.1] when $g_i \equiv 1,\ i = 1, 2, \cdots, n$ and $\tau \equiv 0$. For more details on the periodic solutions of systems (1.1) and (1.2), we refer the readers to [13-16].

The following well-known theorem plays a key role in proving our main results.

Theorem A[22,23] *Let E be a Banach space and P be a cone in E. For $r > 0$, define $P_r = \{u \in P : \|u\| < r\}$. Assume $T : \overline{P}_r \to P$ is completely continuous such that $Tu \ne u$ for $u \in \partial P_r = \{u \in P : \|u\| = r\}$.*

(i) *If* $\|Tu\| > \|u\|$ *for* $u \in \partial P_r$, *then* $i(T, P_r, P) = 0$.

(ii) *If* $\|Tu\| < \|u\|$ *for* $u \in \partial P_r$, *then* $i(T, P_r, P) = 1$.

2 Preliminaries

Set

$$\sigma_{l_i} = \prod_{s=0}^{T-1} (1 - a_i(s) l_i), \quad \sigma_{L_i} = \prod_{s=0}^{T-1} (1 - a_i(s) L_i), \quad \sigma = \min_{i=1,2,\cdots,n} \Big\{ \frac{\sigma_{L_i}(1 - \sigma_{l_i})}{1 - \sigma_{L_i}} \Big\}.$$

For $r > 0$, define

$$M(r) = \max \big\{ f_i(\mathbf{u}) : \mathbf{u} \in \mathbb{R}_+^n,\ \sigma r \le \|\mathbf{u}\| \le r,\ i = 1, 2, \cdots, n \big\} > 0,$$

$$m(r) = \min \big\{ f_i(\mathbf{u}) : \mathbf{u} \in \mathbb{R}_+^n,\ \sigma r \le \|\mathbf{u}\| \le r,\ i = 1, 2, \cdots, n \big\} > 0,$$

$$\Gamma = \sigma \cdot \min_{i=1,2,\cdots,n} \left\{ \sum_{s=0}^{T-1} b_i(s) \frac{\sigma_{L_i}}{1 - \sigma_{L_i}} \right\}, \quad \Lambda = \sum_{i=1}^{n} \left(\sum_{s=0}^{T-1} b_i(s) \frac{1}{1 - \sigma_{l_i}} \right).$$

Let $E = \big\{u : \mathbb{Z} \to \mathbb{R} \,|\, u(t+T) = u(t),\, t \in \mathbb{Z}\big\}$ be a Banach space with the norm $\|u\|_\infty = \max\limits_{t\in\mathbb{T}} |u(t)|$, and X be a Banach space defined by

$$X := \overbrace{E \times E \times \cdots \times E}^{n},$$

which is equipped with the norm $\|\mathbf{u}\| = \sum\limits_{i=1}^{n} \|u_i\|_\infty$ for $\mathbf{u} = (u_1, \cdots, u_n) \in X$.

Define

$$K = \Big\{\mathbf{u} \in X : \ u_i(t) \geq \frac{\sigma_{L_i}(1-\sigma_{l_i})}{1-\sigma_{L_i}} \|u_i\|_\infty,\ i = 1, \cdots, n,\ t \in \mathbb{T}\Big\}.$$

It is not difficult to check that K is a cone in X. For $r > 0$, let

$$\Omega_r = \{\mathbf{u} \in K : \ \|\mathbf{u}\| < r\},$$

then $\partial\Omega_r = \{\mathbf{u} \in K : \ \|\mathbf{u}\| = r\}$.

Let $\mathbf{T}_\lambda : X \to X$ be a mapping with components $(T_\lambda^1, \cdots, T_\lambda^n)$:

$$T_\lambda^i \mathbf{u}(t) = \lambda \sum_{s=t}^{t+T-1} K_i(t,s) b_i(s) f_i(\mathbf{u}(s-\tau(s))),$$

where

$$K_i(t,s) = \frac{\prod\limits_{\theta=s+1}^{t+T-1}(1-a(\theta)g(\mathbf{u}(\theta)))}{1-\prod\limits_{\theta=0}^{T-1}(1-a(\theta)g(\mathbf{u}(\theta)))}, \quad s \in \{t, t+1, \cdots, t+T-1\}.$$

It follows from (C3) that

$$\frac{\sigma_{L_i}}{1-\sigma_{L_i}} \leq K_i(t,s) \leq \frac{1}{1-\sigma_{l_i}}, \quad s \in \{t, t+1, \cdots, t+T-1\}.$$

Moreover, we can easily get

$$0 < \frac{\sigma_{L_i}(1-\sigma_{l_i})}{1-\sigma_{L_i}} < 1.$$

Lemma 2.1 *Let* (C1)-(C3) *hold. Then* $\mathbf{T}_\lambda(K) \subset K$ *and* $\mathbf{T}_\lambda : K \to K$ *is compact and continuous.*

Proof In view of the definition of K, for $\mathbf{u} \in K$ and $i = 1, 2, \cdots, n$,

$$\begin{aligned}(T_\lambda^i \mathbf{u})(t+T) &= \lambda \sum_{s=t+T}^{t+2T-1} K_i(t+T,s) b_i(s) f_i(\mathbf{u}(s-\tau(s))) \\ &= \lambda \sum_{s=t}^{t+T-1} K_i(t+T,s+T) b_i(s+T) f_i(\mathbf{u}(s+T-\tau(s+T))) \\ &= \lambda \sum_{s=t}^{t+T-1} K_i(t+T,s+T) b_i(s) f_i(\mathbf{u}(s-\tau(s))) = (T_\lambda^i \mathbf{u})(t).\end{aligned}$$

Indeed, since a_i is T-periodic and $\mathbf{u} \in K$, we get

$$K_i(t+T, s+T) = K_i(t, s), \quad i = 1, 2, \cdots, n,$$

and thus $\mathbf{T}_\lambda \mathbf{u} \in X$. One can show that, for $\mathbf{u} \in K$ and $t \in \mathbb{T}$,

$$\begin{aligned}(T^i_\lambda \mathbf{u})(t) &\geq \frac{\sigma_{L_i}}{1-\sigma_{L_i}} \sum_{s=t}^{t+T-1} b_i(s) f_i(\mathbf{u}(s-\tau(s))) \\ &= \frac{\sigma_{L_i}(1-\sigma_{l_i})}{1-\sigma_{L_i}} \cdot \frac{1}{1-\sigma_{l_i}} \cdot \lambda \sum_{s=t}^{t+T-1} b_i(s) f_i(\mathbf{u}(s-\tau(s))) \\ &\geq \frac{\sigma_{L_i}(1-\sigma_{l_i})}{1-\sigma_{L_i}} \|T^i_\lambda \mathbf{u}\|_\infty, \quad i = 1, 2, \cdots, n.\end{aligned}$$

Therefore $\mathbf{T}_\lambda(K) \subset K$ and $\mathbf{T}_\lambda : K \to K$ is compact and continuous. The proof is completed.

Using the similar methods as in the proof of [12, Lemma 2.2] with obvious changes, we can obtain the following lemma.

Lemma 2.2 *Let* (C1)-(C3) *hold. Then* $\mathbf{u} \in K\backslash\{\mathbf{0}\}$ *is a positive periodic solution of system* (1.1) *if and only if* $\mathbf{u}$ *is a fixed point of* $\mathbf{T}_\lambda$ *in* $K\backslash\{\mathbf{0}\}$.

Lemma 2.3 *Let* (C1)-(C3) *hold. For any* $\eta > 0$ *and* $\mathbf{u} \in K\backslash\{\mathbf{0}\}$, *if there exists a* f_i *such that* $f_i(\mathbf{u}(t)) \geq \sum\limits_{j=1}^{n} u_j(t)\eta$ *for* $t \in \mathbb{T}$, *then* $\|\mathbf{T}_\lambda \mathbf{u}\| \geq \lambda\Gamma\eta\|\mathbf{u}\|$.

Proof Since $\mathbf{u} \in K\backslash\{\mathbf{0}\}$ and $f_i(\mathbf{u}(t)) \geq \sum\limits_{j=1}^{n} u_j(t)\eta$ for $t \in \mathbb{T}$, we have

$$\begin{aligned}T^i_\lambda \mathbf{u}(t) &\geq \frac{\sigma_{L_i}}{1-\sigma_{L_i}} \lambda \sum_{s=t}^{t+T-1} b_i(s) f_i(\mathbf{u}(s-\tau(s))) \\ &\geq \lambda \frac{\sigma_{L_i}}{1-\sigma_{L_i}} \sum_{s=0}^{T-1} b_i(s) \left(\eta \sum_{j=1}^{n} \frac{\sigma_{L_j}(1-\sigma_{l_j})}{1-\sigma_{L_j}} \|u_j\|_\infty \right) \\ &\geq \lambda\sigma \sum_{s=0}^{T-1} b_i(s) \frac{\sigma_{L_i}}{1-\sigma_{L_i}} \eta \|\mathbf{u}\|,\end{aligned}$$

which implies $\|\mathbf{T}_\lambda \mathbf{u}\| \geq \lambda\Gamma\eta\|\mathbf{u}\|$. The proof is completed.

Let $\widehat{f_i} : [1, \infty) \to \mathbb{R}_+$ be a function defined by

$$\widehat{f_i}(s) = \max\{f_i(\mathbf{u}) : \mathbf{u} \in \mathbb{R}^n_+,\ 1 \leq \|\mathbf{u}\| \leq s\}, \quad i = 1, 2, \cdots, n.$$

Then $\widehat{f_i}$ is nondecreasing on $[1, \infty)$.

Lemma 2.4[11,12] *If* $\lim\limits_{\|\mathbf{u}\|\to\infty} \frac{f_i(\mathbf{u})}{\|\mathbf{u}\|}$ *exists* (*which can be infinity*), *then* $\lim\limits_{s\to\infty} \frac{\widehat{f_i}(s)}{s}$ *exists and*

$$\lim_{s\to\infty}\frac{\widehat{f}_i(s)}{s}=\lim_{\|\mathbf{u}\|\to\infty}\frac{f_i(\mathbf{u})}{\|\mathbf{u}\|}.$$

Lemma 2.5 *Suppose* (C1)-(C3) *hold and $r>\frac{1}{\sigma}$. If there exists an $\varepsilon>0$ such that $\widehat{f}_i(r)\le\varepsilon r,\ i=1,2,\cdots,n$, then $\|\mathbf{T}_\lambda\mathbf{u}\|\le\lambda\varepsilon\Lambda\|\mathbf{u}\|$ for $\mathbf{u}\in\partial\Omega_r$.*

Proof For $\mathbf{u}\in\partial\Omega_r$, we have

$$\begin{aligned}\|\mathbf{T}_\lambda\mathbf{u}\|&\le\lambda\cdot\sum_{i=1}^{n}\sum_{s=t}^{t+T-1}\frac{1}{1-\sigma_{l_i}}b_i(s)\widehat{f}_i(\mathbf{u}(s-\tau(s)))\\&\le\lambda\cdot\sum_{i=1}^{n}\sum_{s=0}^{T-1}\frac{1}{1-\sigma_{l_i}}b_i(s)\varepsilon\|\mathbf{u}\|\\&=\lambda\cdot\sum_{i=1}^{n}\left(\sum_{s=0}^{T-1}b_i(s)\frac{1}{1-\sigma_{l_i}}\right)\cdot\varepsilon\|\mathbf{u}\|=\lambda\Lambda\varepsilon\|\mathbf{u}\|,\end{aligned}$$

and the proof is completed.

When $\mathbf{u}\in\partial\Omega_r,\ r>0$, the definitions of $M(r)$ and $m(r)$ yield

$$m(r)\le f_i(\mathbf{u}(\mathbf{t}))\le M(r),\quad t\in\mathbb{T},\ i=1,2,\cdots,n.$$

Thus by the similar manners as in the proof of Lemmas 2.3 and 2.5, we can easily obtain the following lemmas.

Lemma 2.6 *Let* (C1)-(C3) *hold. If $\mathbf{u}\in\partial\Omega_r$ and $r>0$, then $\|\mathbf{T}_\lambda\mathbf{u}\|\ge\lambda\frac{\Gamma}{\sigma}m(r)$.*

Lemma 2.7 *Let* (C1)-(C3) *hold. If $\mathbf{u}\in\partial\Omega_r$ and $r>0$, then $\|\mathbf{T}_\lambda\mathbf{u}\|\le\lambda\Lambda M(r)$.*

3 Proof of Theorem 1.1

(i) It follows from the assumption that there exists an $r_1>0$ such that

$$f_i(\mathbf{u})\ge\eta\|\mathbf{u}\|$$

for $\mathbf{u}\in\mathbb{R}_+^n$ with $0<\|\mathbf{u}\|\le r_1$, where $\eta>0$ is chosen satisfying $\lambda\Gamma\eta>1$. If $\mathbf{u}\in\partial\Omega_{r_1}$, then

$$f_i(\mathbf{u}(\mathbf{t}))\ge\eta\sum_{j=1}^{n}u_j(t),\quad\text{for } t\in\mathbb{T}.$$

Lemma 2.3 implies $\|\mathbf{T}_\lambda\mathbf{u}\|\ge\lambda\Gamma\eta\|\mathbf{u}\|>\|\mathbf{u}\|$, for $\mathbf{u}\in\partial\Omega_{r_1}$.

On the other hand, since $\lim\limits_{\|\mathbf{u}\|\to\infty}\frac{f_i(\mathbf{u})}{\|\mathbf{u}\|}=0,\ i=1,2,\cdots,n$, Lemma 2.4 yields $\lim\limits_{s\to\infty}\frac{\widehat{f}_i(s)}{s}=0,\ i=1,2,\cdots,n$. Therefore there exists an $r_2>\max\{2r_1,\frac{1}{\sigma}\}$ such that

$$\widehat{f}_i(r_2)\le\varepsilon r_2,\quad i=1,2,\cdots,n,$$

where $\varepsilon>0$ satisfies $\lambda\Lambda\varepsilon<1$. And then by Lemma 2.5, we get

$$\|\mathbf{T}_\lambda\mathbf{u}\|\le\lambda\varepsilon\Lambda\|\mathbf{u}\|<\|\mathbf{u}\|,\quad\text{for }\mathbf{u}\in\partial\Omega_{r_2}.$$

It follows from Theorem A that

$$i(\mathbf{T}_\lambda, \overline{\Omega}_{r_1}, K) = 0, \quad i(\mathbf{T}_\lambda, \overline{\Omega}_{r_2}, K) = 1,$$

consequently $i(\mathbf{T}_\lambda, \overline{\Omega}_{r_2} \backslash \Omega_{r_1}, K) = 1$. Hence, $\mathbf{T}_\lambda$ has a fixed point $\mathbf{u}$ in $\overline{\Omega}_{r_2} \backslash \Omega_{r_1}$, which is just a positive periodic solution of system (1.1).

(ii) Let $r_1 > 0$ be fixed. By Lemma 2.7, there exists a $\lambda_0 > 0$ such that

$$\|\mathbf{T}_\lambda \mathbf{u}\| < \|\mathbf{u}\|, \quad \text{for } \mathbf{u} \in \partial\Omega_{r_1},\ 0 < \lambda < \lambda_0.$$

In view of $\lim\limits_{\|\mathbf{u}\| \to 0} f_i(\mathbf{u}) = \infty$ for some $i = 1, 2, \cdots, n$, there is a positive number $r_2 < r_1$ such that $f_i(\mathbf{u}) \ge \eta\|\mathbf{u}\|$ for $\mathbf{u} \in \mathbb{R}^n_+$ with $0 < \|\mathbf{u}\| \le r_2$, where $\eta > 0$ is chosen so that $\lambda\Gamma\eta > 1$. Then for $\mathbf{u} \in \partial\Omega_{r_2}$, we get

$$f_i(\mathbf{u}(\mathbf{t})) \ge \eta \sum_{j=1}^{n} u_j(t), \quad t \in \mathbb{T}.$$

Lemma 2.3 implies $\|\mathbf{T}_\lambda \mathbf{u}\| \ge \lambda\Gamma\eta\|\mathbf{u}\| > \|\mathbf{u}\|$, for $\mathbf{u} \in \partial\Omega_{r_2}$.

It follows from $\lim\limits_{\|\mathbf{u}\| \to \infty} \frac{f_i(\mathbf{u})}{\|\mathbf{u}\|} = \infty$, $i = 1, 2, \cdots, n$ that there exists an $\widehat{H} > 0$ such that

$$f_i(\mathbf{u}) \ge \eta\|\mathbf{u}\|$$

for $\mathbf{u} \in \mathbb{R}^n_+$ with $\|\mathbf{u}\| \ge \widehat{H}$, where $\eta > 0$ is chosen so that $\lambda\Gamma\eta > 1$. Let $r_3 = \max\{2r_1, \frac{\widehat{H}}{\sigma}\}$. If $\mathbf{u} \in \partial\Omega_{r_3}$, then

$$\min_{t \in \mathbb{T}} \sum_{i=1}^{n} u_i(t) \ge \sigma\|\mathbf{u}\| = \sigma r_3 \ge \widehat{H},$$

which yields

$$f_i(\mathbf{u}(\mathbf{t})) \ge \eta \sum_{j=1}^{n} u_j(t), \quad \text{for } t \in \mathbb{T}.$$

And then Lemma 2.3 shows

$$\|\mathbf{T}_\lambda \mathbf{u}\| \ge \lambda\Gamma\eta\|\mathbf{u}\| > \|\mathbf{u}\|, \quad \text{for } \mathbf{u} \in \partial\Omega_{r_3}.$$

By Theorem A, we can easily obtain

$$i(\mathbf{T}_\lambda, \overline{\Omega}_{r_1}, K) = 1, \;\; i(\mathbf{T}_\lambda, \overline{\Omega}_{r_2}, K) = 0, \;\; i(\mathbf{T}_\lambda, \overline{\Omega}_{r_3}, K) = 0,$$

consequently

$$i(\mathbf{T}_\lambda, \overline{\Omega}_{r_1} \backslash \Omega_{r_2}, K) = 1, \;\; i(\mathbf{T}_\lambda, \overline{\Omega}_{r_3} \backslash \Omega_{r_1}, K) = -1.$$

Hence $\mathbf{T}_\lambda$ has two fixed points lying in $\overline{\Omega}_{r_1} \backslash \Omega_{r_2}$ and $\overline{\Omega}_{r_3} \backslash \Omega_{r_1}$ for $0 < \lambda < \lambda_0$, which are positive periodic solutions of (1.1).

(iii) For a fixed number $r_1 > 0$, Lemma 2.7 implies there exists a $\lambda_0 > 0$ such that

$$\|\mathbf{T}_\lambda \mathbf{u}\| < \|\mathbf{u}\|, \quad \text{for } \mathbf{u} \in \partial\Omega_{r_1},\ 0 < \lambda < \lambda_0.$$

On the other hand, since $\lim\limits_{\|\mathbf{u}\|\to 0} f_i(\mathbf{u}) = \infty$ for some $i = 1, 2, \cdots, n$, there is a positive number $r_2 < r_1$ such that

$$f_i(\mathbf{u}) \geq \eta\|\mathbf{u}\|,$$

for $\mathbf{u} \in \mathbb{R}_+^n$ with $0 < \|\mathbf{u}\| \leq r_2$, where $\eta > 0$ is chosen so that $\lambda\Gamma\eta > 1$. If $\mathbf{u} \in \partial\Omega_{r_2}$, then

$$f_i(\mathbf{u}(\mathbf{t})) \geq \eta \sum_{j=1}^{n} u_j(t), \quad \text{for } t \in \mathbb{T}.$$

It follows from Lemma 2.3 that $\|\mathbf{T}_\lambda \mathbf{u}\| \geq \lambda\Gamma\eta\|\mathbf{u}\| > \|\mathbf{u}\|$, for $\mathbf{u} \in \partial\Omega_{r_2}$.

Using Theorem A again, we can get

$$i(\mathbf{T}_\lambda, \overline{\Omega}_{r_1}, K) = 1, \quad i(\mathbf{T}_\lambda, \overline{\Omega}_{r_2}, K) = 0,$$

so $i(\mathbf{T}_\lambda, \overline{\Omega}_{r_1}\backslash\Omega_{r_2}, K) = 1$. Hence, $\mathbf{T}_\lambda$ has a fixed point $\mathbf{u}$ in $\overline{\Omega}_{r_1}\backslash\Omega_{r_2}$ for $0 < \lambda < \lambda_0$, which is a positive periodic solution of system (1.1). The proof is completed.

4 Positive Periodic Solutions of System (1.2)

In this Section, we shall establish the existence and multiplicity of positive T-periodic solutions of singular discrete system (1.2), that is,

$$\Delta u_i(t) = a_i(t)g_i(\mathbf{u}(t))u_i(t) - \lambda b_i(t)f_i(\mathbf{u}(t-\tau(t))), \quad t \in \mathbb{Z},\ i = 1, \cdots, n,$$

where $\lambda,\ \tau,\ a_i,\ b_i,\ f_i(\mathbf{u}),\ g_i(\mathbf{u})$ satisfy the same assumptions stated for system (1.1). In view of (1.2), we can define an operator $\mathbf{T}_\lambda : X \to X$ with components $(T_\lambda^1, \cdots, T_\lambda^n)$:

$$T_\lambda^i \mathbf{u}(t) = \lambda \sum_{s=t}^{t+T-1} G_i(t,s) b_i(s) f_i(\mathbf{u}(s-\tau(s))),$$

where

$$G_i(t,s) = \frac{\prod\limits_{\theta=s+1}^{t+T-1}(1 + a(\theta)g_i(\mathbf{u}(\theta)))}{\prod\limits_{\theta=0}^{T-1}(1 + a(\theta)g_i(\mathbf{u}(\theta))) - 1}, \quad s \in \{t, t+1, \cdots, t+T-1\}.$$

Clearly, (C1) and (C2) imply for all $t \in \mathbb{T}$ and $i = 1, 2, \cdots, n$,

$$\frac{1}{\rho_{L_i} - 1} \leq G_i(t,s) \leq \frac{\rho_{L_i}}{\rho_{l_i} - 1}, \quad t \leq s \leq t + T - 1$$

and $0 < \frac{\rho_{l_i}-1}{(\rho_{L_i}-1)\rho_{L_i}} < 1$. Here

$$\rho_{l_i} = \prod_{s=0}^{T-1}(1+a_i(s)l_i), \quad \rho_{L_i} = \prod_{s=0}^{T-1}(1+a_i(s)L_i), \quad i = 1, 2, \cdots, n.$$

Define a cone in X by

$$K = \Big\{\mathbf{u} \in X : u_i(t) \geq \frac{\rho_{l_i} - 1}{(\rho_{L_i} - 1)\rho_{L_i}}\|u_i\|_\infty, \, i = 1, \cdots, n, \; t \in \mathbb{T}\Big\}.$$

By the similar arguments as in Sections 2 and 3, we can establish the following theorems.

Theorem 4.1 *Let* (C1) *and* (C2) *hold. Assume* $\lim_{\|\mathbf{u}\|\to 0} f_i(\mathbf{u}) = \infty$ *for some* $i = 1, 2, \cdots, n$.

(i) *If* $\lim_{\|\mathbf{u}\|\to\infty} \frac{f_i(\mathbf{u})}{\|\mathbf{u}\|} = 0$, $i = 1, 2, \cdots, n$, *then for all* $\lambda > 0$, (1.2) *admits a positive periodic solution.*

(ii) *If* $\lim_{\|\mathbf{u}\|\to\infty} \frac{f_i(\mathbf{u})}{\|\mathbf{u}\|} = \infty$, $i = 1, 2, \cdots, n$, *then* (1.2) *admits two positive periodic solutions for* $\lambda > 0$ *sufficiently small.*

(iii) *There exists a* $\lambda_0 > 0$ *such that* (1.2) *admits a positive periodic solution for* $0 < \lambda < \lambda_0$.

Finally, consider discrete systems (1.1) and (1.2) without singularities, that is, we replace (C2) with the following condition.

($\widehat{\text{C}}$2) $g_i \in C(\mathbb{R}^n_+, [0, \infty))$ satisfies $0 < l_i \leq g_i(\mathbf{u}) \leq L_i < \infty$, $f_i : \mathbb{R}^n_+ \to [0, \infty)$ is continuous and $f_i(\mathbf{u}) > 0$ for $\mathbf{u} \in \mathbb{R}^n_+$ with $\mathbf{u} \neq \mathbf{0}$, $i = 1, 2, \cdots, n$.

Then the following two theorems can be established by the similar methods adopted in Sections 2 and 3.

Theorem 4.2 *Let* (C1), ($\widehat{C}$2) *and* (C3) *hold. Assume* $\lim_{\|\mathbf{u}\|\to 0} \frac{f_i(\mathbf{u})}{\|\mathbf{u}\|} = 0$ *for* $i = 1, 2, \cdots, n$.

(i) *If* $\lim_{\|\mathbf{u}\|\to\infty} \frac{f_i(\mathbf{u})}{\|\mathbf{u}\|} = \infty$, $i = 1, 2, \cdots, n$, *then for all* $\lambda > 0$, (1.1) *admits a positive periodic solution.*

(ii) *If* $\lim_{\|\mathbf{u}\|\to\infty} \frac{f_i(\mathbf{u})}{\|\mathbf{u}\|} = 0$, $i = 1, 2, \cdots, n$, *then* (1.1) *admits two positive periodic solutions for* $\lambda > 0$ *sufficiently large.*

(iii) *There exists a* $\lambda_0 > 0$ *such that* (1.1) *admits a positive periodic solution for* $\lambda > \lambda_0$.

Theorem 4.3 *Let* (C1) *and* ($\widehat{C}$2) *hold. Assume* $\lim_{\|\mathbf{u}\|\to 0} \frac{f_i(\mathbf{u})}{\|\mathbf{u}\|} = 0$ *for* $i = 1, 2, \cdots, n$.

(i) *If* $\lim_{\|\mathbf{u}\|\to\infty} \frac{f_i(\mathbf{u})}{\|\mathbf{u}\|} = \infty$, $i = 1, 2, \cdots, n$, *then for all* $\lambda > 0$, (1.2) *admits a positive periodic solution.*

(ii) *If* $\lim\limits_{\|\mathbf{u}\|\to\infty} \frac{f_i(\mathbf{u})}{\|\mathbf{u}\|} = 0$, $i = 1, 2, \cdots, n$, *then* (1.2) *admits two positive periodic solutions for* $\lambda > 0$ *sufficiently large.*

(iii) *There exists a* $\lambda_0 > 0$ *such that* (1.2) *admits a positive periodic solution for* $\lambda > \lambda_0$.

Remark 4.1 Note that Theorems 4.1-4.3 enrich and complement Theorem 1.1. And obviously, Lemma 2.6 is crucial to prove Theorems 4.2-4.3.

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(*edited by Liangwei Huang*)