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THE SYMMETRY DESCRIPTION OF A CLASS OF FRACTIONAL STURM-LIOUVILLE OPERATOR[*][†]

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Abstract

This paper studies the symmetry of a class of fractional Sturm-Liouville differential equations with right and left fractional derivatives. We give the Hermitian boundary condition description of this problem. Furthermore, the density of minimal operator is given. Then the symmetry of this problem is obtained.

Keywords fractional differential operator; differential operator; Sturm-Liouville; density; symmetric operator

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1 Introduction

Recently, fractional differential equations have drawn much attention. It is caused both by the theory of fractional calculus itself and by the applications in various fields of science and engineering such as control, electrochemistry, electromagnetic, porous media, viscoelasticity, etc, for detail, see [1-5].

In this paper, we consider the following fractional Sturm-Liouville operator:

$$(ly)(x) = {}^{c}\mathcal{D}_{1-}^{\alpha} p(x) D_{0+}^{\alpha} y(x) + q(x)y(x), \tag{1.1}$$

acting on the Hilbert space $L^2[0,1]$ with the following boundary condition:

$$\begin{cases} a_{11} I_{0+}^{1-\alpha} y(0) + a_{12} D_{0+}^{\alpha} y(0) + b_{11} I_{0+}^{1-\alpha} y(1) + b_{12} D_{0+}^{\alpha} y(1) = 0, \\ a_{21} I_{0+}^{1-\alpha} y(0) + a_{22} D_{0+}^{\alpha} y(0) + b_{12} I_{0+}^{1-\alpha} y(1) + b_{22} D_{0+}^{\alpha} y(1) = 0, \end{cases} \tag{1.2}$$

where $p(x) \neq 0$, $p(x) \in C^1(0,1)$, $q(x) \in C(0,1)$, $0 < \alpha < 1$ and $a_{ij}, b_{ij} \in \mathbb{R}$ $(i,j = 1,2)$. $I_{0+}^{1-\alpha}$, D_{0+}^{α} and ${}^{c}\mathcal{D}_{1-}^{\alpha}$ are fractional integral operator, fractional derivative and Caputo fractional derivator acting on given functions respectively. Our purpose is

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to find sufficient conditions on the coefficients matrices A and B to guarantee a symmetric operator, where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

We know that, the second-order Sturm-Liouville differential expression

$$ly = -(py')' + qy, \tag{1.3}$$

where $p(x) \neq 0$, $\frac{1}{p}, q \in L(0,1)$, with the following boundary condition:

$$\begin{cases} a_{11}y(0) + a_{12}y'(0) + b_{11}y(1) + b_{12}y'(1) = 0, \\ a_{21}y(0) + a_{22}y'(0) + b_{21}y(1) + b_{22}y'(1) = 0, \end{cases} \tag{1.4}$$

when the coefficient matrices satisfy

$$AQ(0)^{-1}A^* = BQ(1)^{-1}B^*, \tag{1.5}$$

generates a self-adjoint operator, where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad Q(x) = \begin{pmatrix} 0 & -p(x) \\ p(x) & 0 \end{pmatrix}.$$

This result has great significance in the spectral theory of Sturm-Liouville operator.

A nature thought is to imitate the second-order Sturm-Liouville problem, and thereby the self-adjoined description of the fractional Sturm-Liouville problem is given. However, a fractional derivation operator is quite different from an ordinary differential operator in some properties, especially we fail to get the existence and uniqueness of the fractional equation till now. Therefore, the method can not be completely applied to study our problem.

2 Preliminaries

We will use the following properties of fractional integrals and derivatives.

Definition 2.1[3] For given α with $R(\alpha) > 0$, the left and right Riemann-Liouville integrals of order α are defined as

$$(I_{0+}^{\alpha}f)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}f(s)\mathrm{d}s, \quad x \in (0,1],$$
$$(I_{1-}^{\alpha}f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^1 (s-x)^{\alpha-1}f(s)\mathrm{d}s, \quad x \in [0,1).$$

Definition 2.2[3] Let $R(\alpha) \in (n-1,n)$, where $n \in \mathbb{N}$, $D = \frac{\mathrm{d}}{\mathrm{d}x}$. The left and right Riemann-Liouville derivatives of order α are defined as

$$(D_{0+}^{\alpha}f)(x)=D^{n}(I_{0+}^{n-\alpha}f)(x),\quad x\in(0,1],$$
$$(D_{1-}^{\alpha}f)(x)=(-D)^{n}(I_{1-}^{n-\alpha}f)(x),\quad x\in[0,1).$$

In particular, when $0<\alpha<1$, we have

$$(D_{0+}^{\alpha}f)(x)=D(I_{0+}^{1-\alpha}f)(x),\quad x\in(0,1],$$
$$(D_{1-}^{\alpha}f)(x)=(-D)(I_{1-}^{1-\alpha}f)(x),\quad x\in[0,1).$$

Definition 2.3[3] Let $R(\alpha)\in(n-1,n)$, where $n\in\mathbb{N}$, $D=\frac{\mathrm{d}}{\mathrm{d}x}$. The left and right Caputo derivatives of order α are defined as

$$({}^{C}D_{0+}^{\alpha}f)(x)=(I_{0+}^{n-\alpha}D^{n}f)(x),\quad x\in(0,1],$$
$$({}^{C}D_{1-}^{\alpha}f)(x)=(I_{1-}^{n-\alpha}(-D)^{n}f)(x),\quad x\in[0,1).$$

In particular, when $0<\alpha<1$, we have

$$({}^{C}D_{0+}^{\alpha}f)(x)=(I_{0+}^{1-\alpha}Df)(x),\quad x\in(0,1],$$
$$({}^{C}D_{1-}^{\alpha}f)(x)=(I_{1-}^{1-\alpha}(-D)f)(x),\quad x\in[0,1).$$

Lemma 2.1[3] *Let $R(\alpha)\in(n-1,n)$, where $n\in\mathbb{N}$. If $y\in AC^{n}[0,1]$, then the fractional derivatives D_{0+}^{α} and D_{1-}^{α} exist almost everywhere on $[0,1]$ and can be represented in forms*

$$(D_{0+}^{\alpha}y)(x)=\sum_{k=0}^{n-1}\frac{y^{(k)}(0)}{\Gamma(1+k-\alpha)}x^{k-\alpha}+\frac{1}{\Gamma(n-\alpha)}\int_{0}^{x}\frac{y^{(n)}(t)\mathrm{d}t}{(x-t)\alpha-n+1},\quad x\in(0,1],$$
$$(D_{1-}^{\alpha}y)(x)=\sum_{k=0}^{n-1}\frac{(-1)^{k}y^{(k)}(1)}{\Gamma(1+k-\alpha)}(1-x)^{k-\alpha}+\frac{(-1)^{n}}{\Gamma(n-\alpha)}\int_{x}^{1}\frac{y^{(n)}(t)\mathrm{d}t}{(t-x)\alpha-n+1},\quad x\in(0,1].$$

Lemma 2.2[3] *Let $\alpha\in(n-1,n)$, where $n\in\mathbb{N}$. Then operators D_{0+}^{α}, D_{1-}^{α}, ${}^{c}D_{0+}^{\alpha}$ and ${}^{c}D_{1-}^{\alpha}$ satisfy the following integrations by parts,*

$$\int_{0}^{1}f(x)D_{1-}^{\alpha}g(x)\mathrm{d}x=\int_{0}^{1}g(x){}^{c}D_{0+}^{\alpha}f(x)\mathrm{d}x+\sum_{k=0}^{n-1}(-1)^{n-k}f^{(k)}(x)D^{n-k-1}I_{1-}^{n-\alpha}g(x)|_{x=0}^{x=1},$$
$$\int_{0}^{1}f(x)D_{0+}^{\alpha}g(x)\mathrm{d}x=\int_{0}^{1}g(x){}^{c}D_{1-}^{\alpha}f(x)\mathrm{d}x+\sum_{k=0}^{n-1}(-1)^{k}f^{(k)}(x)D^{n-k-1}I_{0+}^{n-\alpha}g(x)|_{x=0}^{x=1}.$$

Lemma 2.3[3] *Fractional integration operators I_{0+}^{α} and I_{1-}^{α} with $R(\alpha)\geq0$ are bounded in $L^{p}(0,1)\,(1\leq p\leq\infty)$:*

$$\parallel I_{0+}^{\alpha}f\parallel_{p}\leq K\parallel f\parallel_{p},\ \parallel D_{1-}^{\alpha}f\parallel_{p}\leq K\parallel f\parallel_{p}\quad with\ K=\frac{1}{R(\alpha)|\Gamma(\alpha)|}.$$

Corollary 2.1 *Fractional integration operators I^{α}_{0+} and I^{α}_{1-} are operators defined on $L^p(0,1)\,(1\le p\le\infty)$.*

Proof For all $f\in L^p(0,1)$, it is easy to see that f satisfies

$$\|f\|_p=\Big(\int_0^1|f|^p\mathrm{d}x\Big)^{\frac{1}{p}}<\infty,$$

so

$$\|I^{\alpha}_{0+}f\|_p=\Big(\int_0^1|I^{\alpha}_{0+}f|^p\mathrm{d}x\Big)^{\frac{1}{p}}\le K\|f\|_p=K\Big(\int_0^1|f|^p\mathrm{d}x\Big)^{\frac{1}{p}}<\infty.$$

Therefore, the integral operator I^{α}_{0+} is an operator from $L^p(0,1)$ to $L^p(0,1)\,(1\le p\le\infty)$.

Lemma 2.4[3] *If $R(\alpha)>0$ and $f(x)\in L^p(0,1)\,(1\le p\le\infty)$, then the following equalities*

$$(D^{\alpha}_{0+}I^{\alpha}_{0+}f)(x)=f(x)\quad \textit{and}\quad (D^{\alpha}_{1-}I^{\alpha}_{1-}f)(x)=f(x)$$

hold almost everywhere on $[0,1]$.

Lemma 2.5[3] *If $\alpha\notin\mathbb{N}$, then the Caputo fractional derivatives coincide with the Riemann-Liouville fractional derivatives in the following cases:*

$$({}^{C}D^{\alpha}_{0+}f)(x)=(D^{\alpha}_{0+}f)(x),$$

if

$$y(0)=y'(0)=\cdots=y^{(n-1)}(0)=0,\quad n=[R(\alpha)]+1;$$

$$({}^{C}D^{\alpha}_{1-}f)(x)=(D^{\alpha}_{1-}f)(x),$$

if

$$y(1)=y'(1)=\cdots=y^{(n-1)}(1)=0,\quad n=[R(\alpha)]+1.$$

Lemma 2.6[3] *The space $AC^n[0,1]$ consists of those and only those functions $f(x)$ which can be represented in the form*

$$f(x)=(I^{n}_{0+}\varphi)(x)+\sum_{k=0}^{n-1}c_kx^k,$$

where $\varphi\in L(0,1)$, $c_k(k=0,1,\cdots,n-1)$ are arbitrary constants, and

$$(I^{n}_{0+}\varphi)(x)=\frac{1}{(n-1)!}\int_0^x(x-t)^{n-1}\varphi(t)\mathrm{d}t.$$

Consider the following fractional differential equation

$$(ly)(x)={}^{c}D^{\alpha}_{1-}p(x)D^{\alpha}_{0+}y(x)+q(x)y(x).\tag{2.1}$$

Assume that $p(x) \neq 0$, $p(x) \in C^1(0,1)$ and $q(x) \in C(0,1)$, obviously, l cannot act on the whole space $L^2[0,1]$.

Definition 2.4 The subset of $L^2[0,1]$ which satisfies:

(1) $D^{\alpha}_{0+}y \in AC[0,1]$,

(2) $ly \in L^2[0,1]$

is called the maximal operator domain of l, denoted by $\mathscr{D}_M$. The operator generated by it is called maximal operator, denoted by $\mathscr{L}_M$.

Definition 2.5 The subset of $\mathscr{D}_M$ which satisfies:

(1) $(I^{1-\alpha}_{0+}y)(x)|_{x=0} = 0$,

(2) $(D^{\alpha}_{0+}y)(x)|_{x=0} = 0$,

(3) $(D^{\alpha}_{0+}y)(x)|_{x=1} = 0$

is called the minimal operator domain of l, denoted by $\mathscr{D}_o$. The operator generated by it is called minimal operator, denoted by $\mathscr{L}_o$.

Let y and z be two functions of $\mathscr{D}_M$. Using Lemma 2.2, we obtain

$$\begin{aligned}(l(y),\overline{z}) &= \int_0^1 l(y)\cdot z\mathrm{d}x = \int_0^1 ({}^cD^{\alpha}_{1-}pD^{\alpha}_{0+}y + qy)\cdot z\mathrm{d}x = \int_0^1 ({}^cD^{\alpha}_{1-}pD^{\alpha}_{0+}y\cdot z + qy\cdot z)\mathrm{d}x \\ &= \int_0^1 pD^{\alpha}_{0+}yD^{\alpha}_{0+}z\mathrm{d}x - pD^{\alpha}_{0+}yI^{1-\alpha}_{0+}z|_0^1 + \int_0^1 qyz\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}(y,l(\overline{z})) &= \int_0^1 y\cdot l(z)\mathrm{d}x = \int_0^1 y\cdot ({}^cD^{\alpha}_{1-}pD^{\alpha}_{0+}z + qz)\mathrm{d}x = \int_0^1 (y\cdot {}^cD^{\alpha}_{1-}pD^{\alpha}_{0+}z + y\cdot qz)\mathrm{d}x \\ &= \int_0^1 pD^{\alpha}_{0+}yD^{\alpha}_{0+}z\mathrm{d}x - pD^{\alpha}_{0+}zI^{1-\alpha}_{0+}y|_0^1 + \int_0^1 qyz\mathrm{d}x.\end{aligned}$$

So,

$$\begin{aligned}(l(y),\overline{z}) - (y,l(\overline{z})) &= \int_0^1 pD^{\alpha}_{0+}yD^{\alpha}_{0+}z\mathrm{d}x - pD^{\alpha}_{0+}yI^{1-\alpha}_{0+}z|_0^1 + \int_0^1 qyz\mathrm{d}x \\ &\quad -\Big(\int_0^1 pD^{\alpha}_{0+}yD^{\alpha}_{0+}z\mathrm{d}x - pD^{\alpha}_{0+}zI^{1-\alpha}_{0+}y|_0^1 + \int_0^1 qyz\mathrm{d}x\Big) \\ &= p(D^{\alpha}_{0+}yI^{1-\alpha}_{0+}z - D^{\alpha}_{0+}zI^{1-\alpha}_{0+}y)|_0^1.\end{aligned}$$

Denote

$$[yz](x) = p(x)[I^{1-\alpha}_{0+}y(x)\cdot D^{\alpha}_{0+}z(x) - I^{1-\alpha}_{0+}z(x)\cdot D^{\alpha}_{0+}y(x)]. \tag{2.2}$$

$[yz](x)$ is called the lagrange bilinear form about l, and it is also called the fit function of y and z. We denote by $[yz]|_0^1 = [yz](1) - [yz](0)$. Then we obtain

$$\int_0^1 l(y)\cdot z\mathrm{d}x - \int_0^1 y\cdot l(z)\mathrm{d}x = [yz]|_0^1. \tag{2.3}$$

Denote

$$C(y)=\begin{pmatrix} I_{0+}^{1-\alpha}y \\ D_{0+}^{\alpha}y \end{pmatrix},\quad C(z)=\begin{pmatrix} I_{0+}^{1-\alpha}z \\ D_{0+}^{\alpha}z \end{pmatrix},\quad Q(x)=\begin{pmatrix} 0 & -p(x) \\ p(x) & 0 \end{pmatrix}.$$

Then

$$[yz](x)=(QC(y),C(z)). \tag{2.4}$$

$Q(x)$ is called the fit matrix of l.

Definition 2.6 Let $y\in L^2[0,1]$, the boundary type of y is defined as

$$U_i(y)=a_{i1}I_{0+}^{1-\alpha}y(0)+a_{i2}D_{0+}^{\alpha}y(0)+b_{i1}I_{0+}^{1-\alpha}y(1)+b_{i2}I_{0+}^{\alpha}y(1), \tag{2.5}$$

where $a_{ij},b_{ij}\in\mathbb{C}$ $(i=1,2,\cdots,r;1\le r\le 4)$.

Denote

$$U(y)=\begin{pmatrix} U_1(y) \\ \vdots \\ U_r(y) \end{pmatrix},\quad A=[a_{ij}],\quad B=[b_{ij}],\quad i=1,2;\ j=1,\cdots,r.$$

So $U(y)$ can be expressed in the following form

$$U(y)=AC(y)_0+BC(y)_1,$$

where $C(y)_0=C(y)|_{x=0}$, and $C(y)_1$ is defined in the same way. We call $U(y)$ the boundary type vector.

Let

$$(A\oplus B)=\begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ \vdots & \vdots & \vdots & \vdots \\ a_{r1} & a_{r2} & b_{r1} & b_{r2} \end{pmatrix},\quad \widetilde{C}(y)=\begin{pmatrix} C(y)_a \\ C(y)_b \end{pmatrix}.$$

Moreover,

$$U(y)=(A\oplus B)\,\widetilde{C}(y). \tag{2.6}$$

If $\operatorname{rank}(A\oplus B)=r$, $U(y)$ is called of r-dimensional, or $U_1,\cdots,U_r$ are linearly independent. We always assume they are linearly independent in the following discussion.

For $r=1,2,3$, let $U_{r+1},\cdots,U_4$ be $4-r$ linearly independent boundary types, and their vector form is U_cy. If the boundary type vector

$$\widetilde{U}(y)=\begin{pmatrix} U(y) \\ U_c(y) \end{pmatrix}$$

is 4-dimensional, we say that $U(y)$ and $U_c(y)$ are complementary. Apparently, it is possible to find complementary boundary type vector of any boundary type vector.

3 Main Results

Theorem 3.1 *For given r-dimensional boundary type vector $U(y)$ and its complementary $(4-r)$-dimensional boundary type vector $U_c(y)$, there is a unique r-dimensional $V_c(z)$ and its $(4-r)$-dimensional boundary type vector $V(z)$, which satisfy*

$$[yz]|_0^1 = U(y)\cdot V_c(z) + U_c(y)\cdot V(z),$$

where "$\cdot$" is the inner product of Euclidean space.

Proof Let

$$U(y) = AC(y)_0 + BC(y)_1 = [A\oplus B]\widetilde{C}(y),$$
$$U_c(y) = \widetilde{A}C(y)_0 + \widetilde{B}C(y)_1 = [\widetilde{A}\oplus\widetilde{B}]\widetilde{C}(y),$$

where A and B are $(r\times 2)$-dimensional matrices, $\widetilde{A}$ and $\widetilde{B}$ are $(4-r)\times 2$-dimensional matrices.

Let

$$H = \begin{pmatrix} A & B \\ \widetilde{A} & \widetilde{B} \end{pmatrix}.$$

Since U and U_c are complementary, H is a non-singular matrix, so we obtain

$$\widetilde{U}(y) = \begin{pmatrix} U(y) \\ U_c(y) \end{pmatrix} = H\widetilde{C}(y), \quad \widetilde{C}(y) = H^{-1}\widetilde{U}(y).$$

By (2.4),

$$[yz]|_0^1 = (Q(1)C(y)_1, C(z)_1) - (Q(0)C(y)_0, C(z)_0).$$

Let

$$\widetilde{Q} = \begin{pmatrix} -Q(0) & 0 \\ 0 & Q(1) \end{pmatrix},$$

then $[yz]|_0^1$ can be changed into

$$[yz]|_0^1 = (\widetilde{Q}\widetilde{C}(y), \widetilde{C}(z)) = (\widetilde{Q}H^{-1}\widetilde{U}(y), \widetilde{C}(z)) = (\widetilde{U}(y), (\widetilde{Q}H^{-1})^*\widetilde{C}(z)).$$

Apparently, $\widetilde{Q}$ is a non-singular matrix. So $(\widetilde{Q}H^{-1})^*$ is a non-singular matrix. Let

$$\widetilde{V}(z) = \begin{pmatrix} V_c(z) \\ V(z) \end{pmatrix} = (\widetilde{Q}H^{-1})^*\widetilde{C}(z),$$

then it is a 4-dimensional boundary type vector, where $V_c(z)$ is an r-dimensional boundary type vector, $V(z)$ is its complementary $(4-r)$-dimensional boundary type vector. Since $\widetilde{V}(z)$ is determined by H,

$$[yz]|_0^1 = U(y)\cdot V_c(z) + U_c(y)\cdot V(z)$$

is determined by U and U_c.

Definition 3.1 $V(z)$ is called the adjoint boundary type vector of $U(y)$, where $V(z)$ and $U(y)$ are defined as above.

If $V(z)$ is an adjoint boundary type vector of $U(y)$, then $U(y)$ is an adjoint boundary type vector of $V(y)$, and they are adjoint boundary type vectors of each other.

Definition 3.2 If two boundary type vectors $V(y)$ and $V'(y)$ satisfy $V'(y) = CV(y)$, where C is a non-singular matrix, then they are called equivalent; if a boundary type vector $V(y)$ is equivalent with its adjoint boundary type vector, we call it is self-conjugate.

Obviously, a boundary type vector is self-conjugate, then its dimension r must satisfy $r = 4 - r$, that is $r = 2$.

Definition 3.3 If $U(y)$ and $V(z)$ are adjoint boundary type vectors of each other, we say that $U(y) = 0$ and $V(z) = 0$ are adjoint boundary conditions of each other; if $U(y)$ is self-conjugate, we say that $U(y) = 0$ is a self-conjugate boundary condition.

Definition 3.4 Assume that H_1 and H_2 are Hilbert spaces, T is an operator from H_1 into H_2, and S is an operator from H_2 into H_1. The operator S is called a formal adjoint of T if we have

$$(g, Tf) = (Sg, f) \quad \text{for any } f \in \mathscr{D}(T),\ g \in \mathscr{D}(S).$$

T is then a formal adjoint of S, too.

Theorem 3.2 *Let $\mathscr{L}$ and $\mathscr{L}'$ be operators generated by $l(y)$. The domain of $\mathscr{L}$ is defined by the boundary condition $U(y) = 0$, and the domain of $\mathscr{L}'$ is defined by the boundary condition $V(y) = 0$. If $U(y) = 0$ and $V(y) = 0$ are adjoint boundary conditions of each other, then $\mathscr{L}$ and $\mathscr{L}'$ are formal adjoints of each other.*

Proof For all $y \in \mathscr{D}(\mathscr{L})$, $z \in \mathscr{D}(\mathscr{L}')$, we have $U(y) = 0$, $V(z) = 0$. Since $U(y) = 0$ and $V(z) = 0$ are adjoint boundary conditions of each other, by Theorem 3.1, there is

$$(l(y), z) - (y, l(z)) = [yz]|_0^1 = U(y) \cdot V_c(z) + U_c(y) \cdot V(z),$$

so we conclude that $(l(y), z) - (y, l(z)) = 0$, that is $(l(y), z) = (y, l(z))$, for any $y \in \mathscr{D}(\mathscr{L})$ and $z \in \mathscr{D}(\mathscr{L}')$. Therefore, $\mathscr{L}$ and $\mathscr{L}'$ are formal adjoints of each other.

Theorem 3.3 *Let the domain of $\mathscr{L}$ be defined by $U(y) = 0$. If $U(y) = 0$ is self-conjugate, then $\mathscr{L}$ is a Hermitian operator.*

Proof For the above $U(y)$ and any of its complementary boundary type vector $U_c(y)$, there exists one and only one $V(z)$ and $V_c(z)$, satisfying

$$[yz]|_0^1 = U(y) \cdot V_c(z) + U_c(y) \cdot V(z).$$

Since $U(y) = 0$ is self-adjoint, there is a non-singular matrix C satisfing $V(z) = CU(Z)$. Therefore,

$$[yz]|_0^1 = U(y) \cdot V_c(z) + U_c(y) \cdot CU(Z).$$

For all $y, z \in \mathscr{D}(\mathscr{L})$, we have $U(y) = V(z) = 0$. Therefore, $(l(y), z) - (y, l(z)) = [yz]|_0^1 = 0$. So

$$(l(y), z) = (y, l(z)) \quad \text{for any } y, z \in \mathscr{D}(\mathscr{L}).$$

We conclude that $\mathscr{L}$ is a Hermitian operator.

Theorem 3.4 *Let $U(y) = AC(y)_0 + BC(y)_1$ and $V(y) = SC(y)_0 + TC(y)_1$ be an r-dimensional and the corresponding $(4-r)$-dimensional boundary type vectors respectively, if*

$$AQ^{-1}(0)S^* = BQ^{-1}(1)T^*,$$

then $U(y) = 0$ and $V(y) = 0$ are adjoint boundary conditions of each other, where $Q(x)$ is a fit matrix of $l(y)$.

Proof We only need to prove that if there are $(4-r)\times 2$ matrices S_1 and T_1 with $\operatorname{rank}(S_1 \oplus T_1) = 4 - r$, which satisfy $AQ(0)^{-1}S_1^* = BQ(1)^{-1}T_1^*$, then we can find a non-singular matrix C satisfying $S_1 = CS$, $T_1 = CT$, thereby

$$V_1(z) = S_1C(z)_0 + T_1C(z)_1$$

is equivalent to $V(z)$. So $U(y) = 0$ and $V(y) = 0$ are adjoint boundary conditions of each other.

Since we assume that $\operatorname{rank}(A\oplus B) = r$, the linear space spanned by r linearly independent row vector of $(A\oplus B)$ is denoted by $\mathscr{T}$. Since

$$-AQ^{-1}(0)S_1^* + BQ^{-1}(1)T_1^* = 0,$$

we have

$$(A\oplus B)\begin{pmatrix} -Q^{-1}(0)S_1^* \\ Q^{-1}(1)T_1^* \end{pmatrix} = 0,$$

which shows that the column vectors of matrix

$$R_1 = \begin{pmatrix} -Q^{-1}(0)S_1^* \\ Q^{-1}(1)T_1^* \end{pmatrix}$$

all belong to $\mathscr{T}^{\perp}$. Since

$$R_1 = \begin{pmatrix} -Q(0) & 0 \\ 0 & Q(1) \end{pmatrix}\begin{pmatrix} S_1^* \\ T_1^* \end{pmatrix} = \widetilde{Q}(S_1\oplus T_1)^*$$

and $\mathrm{rank}(S_1+T_1)=4-r$, we conclude that $\mathrm{rank}R_1=4-r$, which shows that all $(4-r)$ column vectors of R_1 are linearly independent, therefore they make up the basis of $\mathscr{T}^{\perp}$.

Since S, T and S_1, T_1 satisfy the above conditions, we can get that the rank of matrix

$$R=\begin{pmatrix}-Q^{-1}(0)S^*\\ Q^{-1}(1)T^*\end{pmatrix}$$

is $(4-r)$, and all $(4-r)$ column vectors make up the basis of $\mathscr{T}^{\perp}$. Therefore we can find a $(4-r)\times(4-r)$ non-singular matrix C satisfying $S_1=CS$, $T_1=CT$, and this shows that $V_1(z)$ is equivalent to $V(z)$, thereby $U(y)=0$ and $V(y)=0$ are adjoint boundary conditions of each other.

Corollary 3.1 *If the boundary condition* $U(y)=AC(y)_0+BC(y)_1$ *satisfies* $AQ^{-1}(0)A^*=BQ^{-1}(1)B^*$, *then* $U(y)=0$ *define a Hermitian operator.*

Lemma 3.1 $(\widetilde{l}z)(x)={}^cD^{\alpha}_{1-}p(x)Dz(x)$ *is an operator from* $AC^2(0,1)$ *into* $L^2(0,1)$.

Proof The operator

$$(\widetilde{l}z)(x)={}^cD^{\alpha}_{1-}p(x)Dz(x)$$

could be rewritten as

$$(\widetilde{l}z)(x)=(I^{1-\alpha}_{1-}Dp(x)Dz)(x).$$

We known that, for the two-order Sturm-Liouville operator, the maximum operator map $AC^2(0,1)$ into $L^2(0,1)$. So when $z(x)\in AC^2(0,1)$, $(Dp(x)Dz)(x)\in L^2(0,1)$. By Corollary 3.1, $(I^{1-\alpha}_{1-}Dp(x)Dz)(x)\in L^2(0,1)$.

Remark 3.1 By the definition of fractional derivative, $l(y)$ can be rewritten as

$$ly=I^{1-\alpha}_{1-}Dp(x)DI^{1-\alpha}_{0+}y(x)+q(x)y(x). \tag{3.1}$$

In order to make sure that l is an operator on $L^2[0,1]$, we confine that $z(x)=I^{1-\alpha}_{0+}y(x)$ belongs to $AC^2[0,1]$, since we limit $D^{\alpha}_{0+}y\in AC[0,1]$ in the definition of $\mathscr{D}_M$. Based on the above discussion, we describe $\mathscr{D}_M$ in more detail:

$$\mathscr{D}_M=\{y(x)\in L^2(0,1)\,|\,I^{1-\alpha}_{0+}y(x)\in AC^2[0,1]\}. \tag{3.2}$$

Consider a subspace of $\mathscr{D}_M$:

$$\mathscr{E}=\{y(x)\in\mathscr{D}_M\,|\,Ty(x)|_{x=0}=0\}. \tag{3.3}$$

Now we define a new operator:

$$\begin{aligned}T:\quad &\mathscr{E}\longrightarrow AC^2[0,1],\\ &y(x)\longrightarrow I^{1-\alpha}_{0+}y(x).\end{aligned}$$

Theorem 3.5 *T is an invertible operator from $\mathscr{E}$ to $AC^2[0,1]$, further, $T^{-1}=D_{0+}^{1-\alpha}$.*

Proof By Lemma 2.4, when $y(x)\in\mathscr{E}\subseteq L^2(0,1)$,

$$(D_{0+}^{1-\alpha}Ty)(x)=(D_{0+}^{1-\alpha}I_{0+}^{1-\alpha})(x)=y(x),$$

that is $D_{0+}^{1-\alpha}T=I$. When $z(x)\in T\mathscr{E}$,

$$(TD_{0+}^{1-\alpha}z)(x)=(I_{0+}^{1-\alpha}D_{0+}^{1-\alpha}z)(x)=(I_{0+}^{1-\alpha}DI_{0+}^{\alpha}z)(x)=({}^{C}D_{0+}^{\alpha}I_{0+}^{\alpha}z)(x).$$

Since $z(0)=0$, by Lemma 2.4, $({}^{C}D_{0+}^{\alpha}I_{0+}^{\alpha}z)(x)=z(x)$, that is $TD_{0+}^{1-\alpha}=I$. Therefore T is an invertible operator from $\mathscr{E}$ to $AC^2[0,1]$, and $T^{-1}=D_{0+}^{1-\alpha}$.

Lemma 3.2 *Space $AC^2[0,1]$ is a subspace of $AC[0,1]$.*

Proof For all $y(x)\in AC^2[0,1]$, $|y'(x)|$ has a upper bound K, therefore $y(x)$ is a Lipschitz continuous function on $[0,1]$, thus it belongs to $AC[0,1]$.

Corollary 3.2 *$AC[0,1]$ is a dense subspace of $L^2(0,1)$.*

Proof For the second-order Sturm-Liouville operator, the minimum operator domain

$$\widetilde{\mathscr{D}_o}=\{y(x)\in AC^2[0,1]|\,y(x)|_{x=0}=0;y(x)|_{x=1}=0;y'(x)|_{x=0}=0;y'(x)|_{x=1}=0\}$$

is a dense subspace of $L^2(0,1)$, however $\widetilde{\mathscr{D}_o}$ is a subspace of $AC[0,1]$.

Theorem 3.6 *The maximal operator $\mathscr{L}_M$ is a densely defined operator.*

Proof Since $\mathscr{E}$ is a subspace of $\mathscr{D}_M$, it is necessary to prove that $\mathscr{E}$ is a dense subspace of $L^2(0,1)$.

For all $y(x)\in T\mathscr{E}$, by Lemma 2.1, $(D_{0+}^{1-\alpha}y)(x)=(I_{0+}^{\alpha}y')(x)$. Denote $\mathscr{F}=D(T\mathscr{E})$, then $\mathscr{E}=I_{0+}^{\alpha}\mathscr{F}$. By Lemma 2.3, I_{0+}^{α} is a bounded operator on $L^2(0,1)$, thus we only need to prove that $\mathscr{F}$ is a dense subspace of $L^2(0,1)$. In fact, $T\mathscr{E}=\{z(x)\in AC^2[0,1]|\,z(0)=0\}$, by Lemma 2.6, $D(T\mathscr{E})=D(AC^2[0,1])=AC[0,1]$. By Corollary 3.2, it is a dense subset of $L^2(0,1)$.

Theorem 3.7 *The minimal operator $\mathscr{L}_o$ is a densely defined operator.*

Proof According to the definition of Minimal operator,

$$T\mathscr{D}_o=\{z(x)\in AC^2[0,1]|\,z(x)|_{x=0}=0;z'(x)|_{x=0}=0;z'(x)|_{x=1}=0\}.$$

thus

$$\begin{aligned}\mathscr{D}_o&=T^{-1}\{z(x)\in AC^2[0,1]|\,z(x)|_{x=0}=0;z'(x)|_{x=0}=0;z'(x)|_{x=1}=0\}\\&=D_{0+}^{1-\alpha}\{z(x)\in AC^2[0,1]|\,z(x)|_{x=0}=0;z'(x)|_{x=0}=0;z'(x)|_{x=1}=0\}.\end{aligned}$$

Similarly to the above discussion, we obtain

$$\begin{aligned}&D_{0+}^{1-\alpha}\{z(x)\in AC^2[0,1]|\,z(x)|_{x=0}=0;z'(x)|_{x=0}=0;z'(x)|_{x=1}=0\}\\=&I_{0+}^{\alpha}D\{z(x)\in AC^2[0,1]|\,z(x)|_{x=0}=0;z'(x)|_{x=0}=0;z'(x)|_{x=1}=0\}\\=&I_{0+}^{\alpha}\{z(x)\in AC[0,1]|\,z(x)|_{x=0}=0;z(x)|_{x=1}=0\}.\end{aligned}$$

By Corollary 3.2,

$$\{z(x) \in AC[0,1] |\, z(x)|_{x=0} = 0; z(x)|_{x=1} = 0\}$$

is a dense subspace of $L^2(0,1)$. Combined with the boundedness of I^{α}_{0+}, we finish the proof.

Remark 3.2 If we confine $I^{1-\alpha}_{0+} y(x)|_{x=1} = 0$ in addition, that is

$$\frac{1}{\Gamma(1-\alpha)} \int_0^1 (1-s)^{-\alpha} y(s) \mathrm{d}s = 0.$$

When $\alpha \in (0, \frac{1}{2})$, $y(x) \in (\frac{1}{1-x})^{\perp}$. These functions can not be dense in $L^2[0,1]$, since $I^{1-\alpha}_{0+} y(x)|_{x=1} = 0$ does not hold in the definition of minimal operator.

Theorem 3.8 *The boundary conditions of* $l(y)$*:*

$$\begin{cases} a_{11} I^{1-\alpha}_{0+} y(0) + a_{12} D^{\alpha}_{0+} y(0) + b_{12} D^{\alpha}_{0+} y(1) = 0, \\ a_{21} I^{1-\alpha}_{0+} y(0) + a_{22} D^{\alpha}_{0+} y(0) + b_{22} D^{\alpha}_{0+} y(1) = 0 \end{cases} \tag{3.4}$$

satisfing

$$AQ(0)^{-1} A^* = B_0 Q(1)^{-1} B_0^*, \tag{3.5}$$

generate a symmetry operator by l*, where*

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 & b_{12} \\ 0 & b_{22} \end{pmatrix}.$$

Proof Since the minimal operator $\mathscr{L}_o$ is a densely defined operator, that is,

$$\mathscr{D}_o = \{y(x) \in \mathscr{D}_M |\, (I^{1-\alpha}_{0+} y)(x)|_{x=0} = 0; (D^{\alpha}_{0+} y)(x)|_{x=0} = 0; (D^{\alpha}_{0+} y)(x)|_{x=1} = 0\}$$

is a dense subspace of $L^2(0,1)$, therefore the functions with

$$\begin{cases} a_{11} I^{1-\alpha}_{0+} y(0) + a_{12} D^{\alpha}_{0+} y(0) + b_{12} D^{\alpha}_{0+} y(1) = 0, \\ a_{21} I^{1-\alpha}_{0+} y(0) + a_{22} D^{\alpha}_{0+} y(0) + b_{22} D^{\alpha}_{0+} y(1) = 0, \\ a_{31} I^{1-\alpha}_{0+} y(0) + a_{32} D^{\alpha}_{0+} y(0) + b_{32} D^{\alpha}_{0+} y(1) = 0 \end{cases}$$

are dense in $L^2(0,1)$, thus the functions with

$$\begin{cases} a_{11} I^{1-\alpha}_{0+} y(0) + b_{11} D^{\alpha}_{0+} y(0) + b_{12} D^{\alpha}_{0+} y(1) = 0, \\ a_{21} I^{1-\alpha}_{0+} y(0) + b_{21} D^{\alpha}_{0+} y(0) + b_{22} D^{\alpha}_{0+} y(1) = 0 \end{cases}$$

are also dense in $L^2(0,1)$.

By Theorems 3.3 and 3.4, if

$$AQ(0)^{-1} A^* = B_0 Q(1)^{-1} B_0^*,$$

the operator with the above boundary condition is symmetry.

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