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NONEXISTENCE OF POSITIVE SOLUTIONS FOR A FOUR-POINT BOUNDARY VALUE PROBLEM FOR FRACTIONAL DIFFERENTIAL EQUATION*†

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Abstract

In this paper, we investigate the nonexistence of positive solutions for a class of four-point boundary value problem of nonlinear differential equation with fractional order derivative. We give sufficient conditions on nonlinear term and the parameter such that the boundary value problem has no positive solutions. Some examples are presented to illustrate the main results.

Keywords positive solution; fractional differential equation; fixed point; cone

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1 Introduction

In this paper, we consider the nonexistence of the positive solution for the following boundary value problem of differential equation involving the Caputo's fractional order derivative

$$D_{0+}^{\alpha}u(t) + \lambda f(t, u(t)) = 0, \quad t \in (0, 1), \tag{1.1}$$

$$u'(0) - \mu_1 u(\xi) = 0, \quad u'(1) + \mu_2 u(\eta) = 0, \tag{1.2}$$

where $1 < \alpha \leq 2,\ 0 \leq \xi \leq \eta \leq 1,\ 0 \leq \mu_1, \mu_2 \leq 1$ and satisfy the following conditions:

(H1) $\Delta = \mu_1(1 + \mu_2\eta - \mu_2\xi) + \mu_2 < (\alpha - 1)(1 - \mu_1\xi)$;

(H2) $f \in C([0,\ 1] \times R^+, R^+)$.

Due to the development of the theory of fractional calculus and its applications, such as in the fields of physics, electro-dynamics of complex medium, control theory,

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Bode's analysis of feedback amplifiers, blood flow phenomena, aerodynamics and polymer rheology, electron-analytical chemistry, etc, many works on fractional calculus, fractional order differential equations have appeared [1-7]. Recently, there have been many results concerning the solutions and positive solutions for boundary value problems for nonlinear fractional differential equations, see [8-29] and references therein.

For example, Bai and Lü [12] considered the following Dirichlet boundary value problem of fractional differential equation

$$D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, \quad t \in (0,1), \tag{1.3}$$

$$u(0) = 0 = u(1), \quad 1 < \alpha \leq 2. \tag{1.4}$$

By means of different fixed-point theorems on a cone, some existence and multiplicity results of positive solutions were obtained. Jiang and Yuan [20] improved the results in [12] by discussing some new positive properties of the Green function for problem (1.3). By using the fixed point theorem on a cone due to Krasnoselskii, the authors established the existence results of positive solution for problem (1.3). Recently, Caballero et al. [21] obtained the existence and uniqueness of positive solution for singular boundary value problem (1.3). The existence results were established in the case that the nonlinear term f may be singular at $t = 0$.

There are also some results concerning multi-point boundary value problems for differential equations of fractional order.

Wang et al. [25] considered the boundary value problem of fractional differential equation with integral condition

$$D_{0+}^{\alpha}u(t) + q(t)f(t, u(t)) = 0, \quad t \in (0,1), \; n-1 < \alpha < n, \tag{1.5}$$

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(1) = \int_0^1 u(s)\mathrm{d}A(s), \tag{1.6}$$

where $\alpha > 2$, $\int_0^1 u(s)\mathrm{d}A(s)$ was given by Riemann-Stieltjes integral with a signed measure. By using the fixed point theorem, the existence of positive solution for this problem were established.

Many works deal with the existence and multiplicity of positive solution for fractional differential equation (1.1) under the boundary conditions (1.2). Zhao, Chai and Ge [28] considered a class of four-point fractional boundary value problem of the form

$$D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, \quad t \in (0,1), \tag{1.7}$$

$$u'(0) - \mu_1 u(\xi) = 0, \quad u'(1) + \mu_2 u(\eta) = 0, \tag{1.8}$$

where $1 < \alpha \leq 2$, $0 \leq \xi \leq \eta \leq 1$, $0 \leq \mu_1, \mu_2 \leq 1$ with the condition

$$\Delta = \mu_1(1+\mu_2\eta-\mu_2\xi)+\mu_2 < (\alpha-1)(1-\mu_1\xi), \tag{1.9}$$

and $f:[0,1]\times R^+\to R^+$ is continuous. By using fixed-point theorems and successive iteration method, the authors established the existence results of at least one positive solution for this problem.

Yang and Zhang [29] considered the positive solution for the following boundary value problem of differential equation involving the Caputo's fractional order derivative

$$D^\alpha_{0+}u(t)+f(t,u(t),u'(t))=0, \quad t\in(0,1), \tag{1.10}$$

$$u'(0)-\mu_1u(\xi)=0, \quad u'(1)+\mu_2u(\eta)=0, \tag{1.11}$$

where $1<\alpha\le 2,\ 0\le\xi\le\eta\le 1,\ 0\le\mu_1,\mu_2\le 1$. By using the Avery-Peterson fixed point theorem, the existence of at least three positive solutions were established.

To complement the work on the positive solutions of problem (1.1) with (1.2), in this paper we consider the nonexistence of positive solution for problem (1.1) with (1.2). Sufficient conditions on the nonlinear term f and the explicit ranges of parameter λ, under which problem (1.1) with (1.2) has no positive solution, are given in Section 3. Some examples are presented in Section 4 to illustrate the main results.

2 Preliminary Results

Definition 2.1 The Riemann-Liouville fractional integral of order $\alpha>0$ of a function $u:(0,\infty)\to R$ is given by

$$I^\alpha_{0+}u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)\mathrm{d}s,$$

provided the right side is point-wise defined on $(0,\infty)$.

Definition 2.2 The Caputo's fractional derivative of order $\alpha>0$ of a continuous function $u:(0,\infty)\to R$ is given by

$$D^\alpha_{0+}u(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{u^{(n)}(s)}{(t-s)^{\alpha-n+1}}\mathrm{d}s,$$

where $n-1<\alpha\le n$, provided that the right side is point-wise defined on $(0,\infty)$.

Lemma 2.1 *Let $\alpha>0$. Then*

$$I^\alpha_{0+}D^\alpha_{0+}u(t)=u(t)+c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1}, \quad c_i\in R,\ i=1,2,\cdots,n-1.$$

3 Main Results

Lemma 3.1[28] *Given $y(t)\in C[0,1]$. Then following FBVPs*

$$D^\alpha_{0+}u(t)+y(t)=0, \quad t\in(0,1), \tag{3.1}$$

$$u'(0)-\mu_1u(\xi)=0, \quad u'(1)+\mu_2u(\eta)=0 \tag{3.2}$$

is equivalent to an operator equation

$$u(t) = \int_0^1 G(t,\ s)y(s)\mathrm{d}s, \tag{3.3}$$

where

$$G(t,s) = \begin{cases} -\dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \dfrac{\mu_1(1+\mu_2\eta-\mu_2 t)(\xi-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} \\ +\dfrac{\mu_1(1-\mu_1\xi+\mu_1 t)(1-s)^{\alpha-2}}{\Delta\Gamma(\alpha-1)} + \dfrac{\mu_2(1-\mu_1\xi+\mu_1 t)(\eta-s)^{\alpha-1}}{\Delta\Gamma(\alpha)}, & s \le \xi,\ s \le t, \\ \dfrac{\mu_1(1+\mu_2\eta-\mu_2 t)(\xi-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} \\ +\dfrac{\mu_1(1-\mu_1\xi+\mu_1 t)(1-s)^{\alpha-2}}{\Delta\Gamma(\alpha-1)} + \dfrac{\mu_2(1-\mu_1\xi+\mu_1 t)(\eta-s)^{\alpha-1}}{\Delta\Gamma(\alpha)}, & s \le \xi,\ s \ge t, \\ -\dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \dfrac{\mu_1(1-\mu_1\xi+\mu_1 t)(1-s)^{\alpha-2}}{\Delta\Gamma(\alpha-1)} \\ +\dfrac{\mu_2(1-\mu_1\xi+\mu_1 t)(\eta-s)^{\alpha-1}}{\Delta\Gamma(\alpha)}, & \xi \le s \le \eta,\ s \le t, \\ \dfrac{\mu_1(1-\mu_1\xi+\mu_1 t)(1-s)^{\alpha-2}}{\Delta\Gamma(\alpha-1)} + \dfrac{\mu_2(1-\mu_1\xi+\mu_1 t)(\eta-s)^{\alpha-1}}{\Delta\Gamma(\alpha)}, & \xi \le s \le \eta,\ s \ge t, \\ -\dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \dfrac{\mu_1(1-\mu_1\xi+\mu_1 t)(1-s)^{\alpha-2}}{\Delta\Gamma(\alpha-1)}, & s \ge \eta,\ s \le t, \\ \dfrac{\mu_1(1-\mu_1\xi+\mu_1 t)(1-s)^{\alpha-2}}{\Delta\Gamma(\alpha-1)}, & s \ge \eta,\ s \ge t. \end{cases}$$

Lemma 3.2[28] *Let $G(t,s)$ be given as in the statement of Lemma* 3.1. *Then we find that*

(1) *$G(t,s)$ is a continuous function on the unit square* $[0,1]\times[0,1]$;

(2) *$G(t,s) \ge 0$ for each* $(t,s) \in [0,1]\times[0,1]$;

(3) $G(t,s) \le M(1-s)^{\alpha-2},\ s \in (0,1)$;

(4) *there is a positive constant $\gamma_0 \in (0,1)$ such that*

$$\min_{t\in[\frac14,\frac34]} G(t,\ s) \ge \gamma_0 M(1-s)^{\alpha-2}, \quad s \in (0,1), \tag{3.4}$$

where

$$M = \frac{\mu_1(1+\mu_2\eta)+(\alpha-1)(1-\mu_1\xi+\mu_1)+\mu_2(1-\mu_1\xi+\mu_1)}{\Delta\Gamma(\alpha)} > 0,$$

$$\gamma_0 = \frac{(\alpha-1)(1-\mu_1\xi+\frac14\mu_1)-\Delta}{\mu_1(1+\mu_2\eta)+(\alpha-1)(1-\mu_1\xi+\mu_1)+\mu_2(1-\mu_1\xi+\mu_1)} > 0.$$

Here we introduce the following extreme limits:

$$f_0^s = \limsup_{u\to 0^+} \max_{t\in[0,1]} \frac{f(t,u)}{u}, \quad f_0^i = \liminf_{u\to 0^+} \min_{t\in[\frac{1}{4},\frac{3}{4}]} \frac{f(t,u)}{u},$$

$$f_\infty^s = \limsup_{u\to \infty} \max_{t\in[0,1]} \frac{f(t,u)}{u}, \quad f_\infty^i = \liminf_{u\to \infty} \min_{t\in[\frac{1}{4},\frac{3}{4}]} \frac{f(t,u)}{u}.$$

Let the Banach space $E = C[0,1]$ be endowed with the norm

$$\|u\| = \max_{0\le t\le 1} |u(t)|, \quad u \in E.$$

We define a cone $K \subset E$ by

$$K = \{u \in E|\, u(t) \ge 0, \inf_{\frac{1}{4}\le t\le\frac{3}{4}} u(t) \ge \gamma_0\|u(t)\|\}.$$

Lemma 3.3 *Let $T:\ K \to E$ be an operator defined by*

$$Tu(t) := \lambda \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s.$$

Then $T : K \to K$ is completely continuous.

Proof The operator $T : K \to E$ is continuous in view of the continuity of the functions $G(t,s)$ and $f(t,u(t),u'(t))$. Let $\Omega \subset K$ be bounded. Then there exists a positive constant $R_1 > 0$ such that $\|u\| \le R_1,\ u \in \Omega$.

Denote

$$R = \max_{0\le t\le 1,\, u\in\Omega} |f(t,u(t))| + 1.$$

Then for $u \in \Omega$, we have

$$|Tu| \le \lambda \int_0^1 G(t,s)|f(s,u(s))|\mathrm{d}s \le \lambda R \int_0^1 M(1-s)^{\alpha-2}\mathrm{d}s = \frac{\lambda MR}{\alpha-1}.$$

Hence $T(\Omega)$ is bounded. For $u \in \Omega,\ t_1,t_2 \in [0,\ 1]$, one has

$$\begin{aligned}|Tu(t_2) - Tu(t_1)| \le{}& \Big|\frac{\lambda}{\Gamma(\alpha)}\Big(\int_0^{t_2}(t_2-s)^{\alpha-1}f_1(s,u)\mathrm{d}s - \int_0^{t_1}(t_1-s)^{\alpha-1}f(s,u)\mathrm{d}s\Big)\Big|\\ &+\frac{\lambda\mu_1\mu_2}{\Delta\Gamma(\alpha)}\int_0^{\xi}(\xi-s)^{\alpha-1}f(s,u(s))\mathrm{d}s \times |t_2-t_1|\\ &+\frac{\lambda\mu_1}{\Delta\Gamma(\alpha-1)}\int_0^{1}(1-s)^{\alpha-2}f(s,u(s))\mathrm{d}s \times |t_2-t_1|\\ &+\frac{\lambda\mu_1\mu_2}{\Delta\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}f(s,u(s))\mathrm{d}s \times |t_2-t_1|\\ \le{}& \frac{\lambda|t_2^\alpha - t_1^\alpha|}{\Gamma(\alpha+1)} + \frac{\lambda R}{\Delta\Gamma(\alpha+1)}(\mu_1\mu_2(\xi^\alpha+\eta^\alpha)+\alpha\mu_1) \times |t_2-t_1|.\end{aligned}$$

Thus,

$$\|Tu(t_2) - Tu(t_1)\| \to 0 \quad \text{for } t_1 \to t_2,\ u \in \Omega.$$

By means of the Arzela-Ascoli theorem, we claim that T is completely continuous. Finally, we see that

$$\begin{aligned}
\min_{\frac{1}{4} \le t \le \frac{3}{4}} |Tu(t)| &= \min_{\frac{1}{4} \le t \le \frac{3}{4}} \int_0^1 G(t,s) f(s,u(s)) \mathrm{d}s \\
&\ge \gamma_0 \int_0^1 M(1-s)^{\alpha-2} f(s,u(s)) \mathrm{d}s \\
&\ge \gamma_0 \max_{0 \le t \le 1} \int_0^1 G(t,s) f(s,u(s)) \mathrm{d}s \\
&= \gamma_0 \max_{0 \le t \le 1} (Tu)(t).
\end{aligned}$$

Thus, we show that $T : K \to K$ is a completely continuous operator.

Theorem 3.1 *If $f_0^s, f_\infty^s < \infty$, then there exists a positive constant λ_0 such that for every $\lambda \in (0, \lambda_0)$, the boundary value problem* (1.1) *with* (1.2) *has no positive solution.*

Proof From the definitions of f_0^s, f_∞^s and the condition $f_0^s, f_\infty^s < \infty$, there exists an $M_1 > 0$ such that

$$f(t,u) \le M_1 u, \quad t \in [0,1],\ u \ge 0.$$

Define a positive constant

$$\lambda_0 = \frac{\alpha - 1}{M M_1}.$$

Let $\lambda \in (0, \lambda_0)$, then we suppose that problem (1.1) with (1.2) has a positive solution $u(t),\ t \in [0,1]$. Thus,

$$\begin{aligned}
u(t) = Tu(t) = \lambda \int_0^1 G(t,s) f(s,u(s)) \mathrm{d}s &\le \lambda M_1 \int_0^1 M(1-s)^{\alpha-2} u(s) \mathrm{d}s \\
&\le \lambda M_1 \int_0^1 M(1-s)^{\alpha-2} \|u\| \mathrm{d}s = \lambda M_1 M \frac{1}{\alpha - 1} \|u\|.
\end{aligned}$$

Therefore,

$$\|u\| \le \lambda \frac{M M_1}{\alpha - 1} \|u\| < \lambda_0 \frac{M M_1}{\alpha - 1} \|u\| = \|u\|,$$

which is a contradiction. So the boundary value problem (1.1) with (1.2) has no positive solution.

Theorem 3.2 *If $f_0^i, f_\infty^i > 0$, then there exists a positive constant $\widetilde{\lambda}_0$ such that for every $\lambda > \widetilde{\lambda}_0$, the boundary value problem* (1.1) *with* (1.2) *has no positive solution.*

Proof From the definitions of f_0^i and f_∞^i, there exists a positive number m_1 such that

$$f(t,u) \geq m_1 u, \quad t \in [0,1],\ u \geq 0.$$

Define a positive constant

$$\widetilde{\lambda}_0 = \frac{\alpha - 1}{\gamma_0 M m_1}.$$

Let $\lambda > \widetilde{\lambda}_0$, and suppose that problem (1.1) with (1.2) has a positive solution $u(t),\ t \in [0,1]$. Then for $t \in \left[\frac{1}{4}, \frac{3}{4}\right]$, we have

$$\begin{aligned} u(t) = Tu(t) = \lambda \int_0^1 G(t,s) f(s,u(s)) \mathrm{d}s &\geq \lambda \int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s) f(s,u(s)) \mathrm{d}s \\ &\geq \lambda m_1 \gamma_0 \int_{\frac{1}{4}}^{\frac{3}{4}} M(1-s)^{\alpha-2} u \mathrm{d}s \geq \lambda m_1 \gamma_0 M \frac{1}{\alpha - 1} \|u\|. \end{aligned}$$

Thus,

$$\|u\| \geq \lambda m_1 \gamma_0 \frac{M}{\alpha - 1} \|u\| > \widetilde{\lambda}_0 m_1 \gamma_0 \frac{M}{\alpha - 1} \|u\| = \|u\|,$$

which is a contradiction. So the boundary value problem (1.1) with (1.2) has no positive solution.

4 Example

Consider a nonlinear FBVPs

$$ {}^C D_{0+}^{\alpha} u(t) + \lambda f(t, u(t)) = 0, \quad t \in (0,1), \tag{4.1}$$

$$ u'(0) - \mu_1 u(\xi) = 0, \quad u'(1) + \mu_2 u(\eta) = 0, \tag{4.2}$$

where $\alpha = 1.8,\ \mu_1 = \frac{1}{6},\ \mu_2 = \frac{1}{3},\ \xi = \frac{1}{3},\ \eta = \frac{2}{3}$,

$$\begin{aligned} \Delta &= \mu_1(1 + \mu_2\eta - \mu_2\xi) + \mu_2 < (\alpha - 1)(1 - \mu_1\xi) = \frac{14}{27} < \frac{34}{45} = (\alpha - 1)(1 - \mu_1\xi), \\ M &= \frac{\mu_1(1 + \mu_2\eta) + (\alpha - 1)(1 - \mu_1\xi + \mu_1) + \mu_2(1 - \mu_1\xi + \mu_1)}{\Delta\Gamma(\alpha)} = \frac{79}{28\Gamma(1.8)} \approx 3.0293, \\ \gamma_0 &= \frac{(\alpha - 1)(1 - \mu_1\xi + \frac{1}{4}\mu_1) - \Delta}{\mu_1(1 + \mu_2\eta) + (\alpha - 1)(1 - \mu_1\xi + \mu_1) + \mu_2(1 - \mu_1\xi + \mu_1)} = \frac{73}{395} \end{aligned}$$

and

$$f(t,\ u) = \frac{t^{\frac{1}{5}}(40u + 1)(20 + \sin t)u}{u + 1}.$$

By simple computation, we have

$$f_0^s = 20, \quad f_\infty^s = 840, \quad M_1 = 840, \quad f_0^i = \frac{20}{4^{\frac{1}{5}}}, \quad f_\infty^i = \frac{760}{4^{\frac{1}{5}}}, \quad m_1 = \frac{20}{4^{\frac{1}{5}}},$$

$$\lambda_0 = \frac{\alpha - 1}{MM_1} \approx 2.9676 * 10^{-4}, \quad \widetilde{\lambda}_0 = \frac{\alpha - 1}{\gamma_0 M m_1} \approx 0.0943.$$

From Theorem 3.1, for every $\lambda \in (0, \lambda_0)$, problem (4.1) with (4.2) has no positive solution. From Theorem 3.2, for every $\lambda > \widetilde{\lambda}_0$, problem (4.1) with (4.2) has no positive solution.

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