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SOME MATHEMATICAL MODELS AND MATHEMATICAL ANALYSIS ABOUT RAYLEIGH-TAYLOR INSTABILITY*

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Abstract

In this paper we will review some mathematical models and mathematical analysis about Rayleigh-Taylor instability.

Keywords mathematical models; mathematical analysis; Rayleigh-Taylor instability

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1 Introduction

The instability of the interface between two different densities of fluid under the action of gravity or inertial force, as early as 1950, was clearly pointed out by G.L. Taylor, and is often named after him, actually earlier than him, L. Rayleigh in 1900 and S.H. Lamb in 1932 also talked about this problem in some sense, people sometimes call Rayleigh-Taylor or Rayleigh-Lamb-Taylor instability. This interfacial instability phenomenon can be found not only in astrophysics, but also in laster fusion and high-speed collision. It is very important even for hydraulic machinery and various engines. The linear development stage of interface instability is relatively clear. However, there are still many problems in nonlinear development that need to be recognized. The relevant research has very important practical and theoretical value.

In the mathematical analysis theory, since 2003, there have been some breakthrough works on the RT instability of compressible fluids [1,2], free boundary prob-

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lems [3] and MHD fluids [4]. Of course, these theoretical results are still far away form the actual physical mechanics. There is still a big gap in the problem.

2 Some RT Instability Mathematical Models

2.1 Double infinite fluid Taylor instability

We consider the two-layer infinite fluid shown in Figure 2.1, where each of densities ρ_1 and ρ_2 occupies a half plane of $y > 0$ and $y < 0$, and gravity $\vec{g}$ is parallel to y axis, pointing to its negative direction, that is $\vec{g} = -g\vec{j}$.

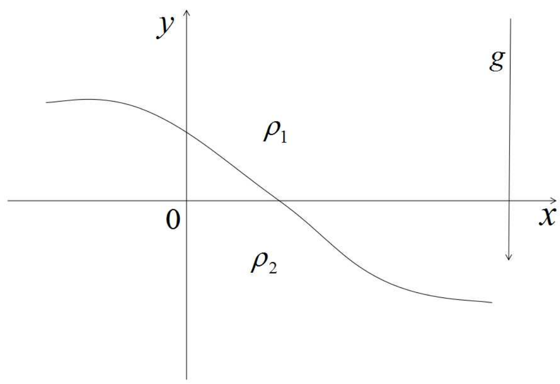

Figure 2.1

The deviation of the initial position of the interface from $y = 0$ is a small amount, that is $y = \varepsilon \cos kx$.

Assume that the two-layer fluid is in a static state at the initial moment, and thus is initially non-rotating. For an ideal incompressible fluid, the fluid remains free-curl in the subsequent movement. Thus, the velocity potential Φ_i $(i = 1, 2)$ can be introduced corresponding to the upper layer and lower layer of fluid. Then under two-dimensional conditions, Φ_i $(i = 1, 2)$ satisfies the Laplace equation

$$\Phi_{ixx} + \Phi_{iyy} = 0, \quad i = 1, 2. \tag{2.1}$$

If we suppose that the interface position during the movement is $y = \eta(x, t)$. And its derivative is the first-order small amount of ε, then

$$F(r, t) = y - \eta(x, t) = 0. \tag{2.2}$$

Therefore,

$$\begin{aligned} &\frac{\partial F}{\partial t} = -\eta_t \sim \varepsilon, \\ &\frac{\partial F}{\partial \eta} = 1, \quad \frac{\partial F}{\partial x} = -\eta_x \sim \varepsilon, \\ &\nabla F = -\eta_x \vec{i} + \vec{j}. \end{aligned} \tag{2.3}$$

The normal vector of the interface is $\vec{n} = \frac{\nabla F}{|\nabla F|}$, and the normal velocity of the interface is $D_n = -\frac{\partial F/\partial t}{|\nabla F|}$. The following subscripts $(i = 1, 2)$ respectively denote the physical quantities in the flow on two sides of the interface, and the conditions of equal pressure and normal velocity on both sides of the fluid interface are continuous

$$p_1 = p_2, \quad D_n = q_1 \cdot n = q_2 \cdot n, \tag{2.4}$$

where p is the pressure and q is the velocity.

On the interface $F(r, t) = 0$,

$$-\frac{\partial F}{\partial t} = \nabla\Phi_1 \cdot \nabla F = \nabla\Phi_2 \cdot \nabla F. \tag{2.5}$$

In the case of potenial, the momentum equation can be integrated

$$\Phi_t + \frac{1}{2}(\nabla\Phi)^2 + \frac{P}{\rho} + U(r) = f'(t), \tag{2.6}$$

where $U(r)$ is the potential of mass power, $f'(t)$ is an arbitrary function. In (2.6), we can choose $\rho_1 f_1(t) = \rho_2 f_2(t)$, then under the condition of retaining a first-order small amount, on $y = \eta(x, t)$, there are

$$\eta_t = \Phi_{1y} = \Phi_{2y}, \quad \rho_1(\Phi_{1t} + g\eta(x, t)) = \rho_2(\Phi_{2t} + g\eta(x, t)). \tag{2.7}$$

Since η is a first-order small quantity, Φ_t is also a first-order small quantity, and the relevant quantity on $y = \eta$ can be performed by Taylor expands, that is

$$\Phi_i|_{y=\eta} = \Phi_i|_{y=0} + \eta\frac{\partial\Phi_i}{\partial y} = \Phi_i|_{y=0} + O(\varepsilon^2). \tag{2.8}$$

That is, the relevant derivative of (2.7) can be expanded by the value of $y = 0$, which implies (2.7) still holds on the unknown active boundary $y = \eta$, but it is also fixed on the fixed boundary $y = 0$ after linearization. This greatly simplifies the solution of the problem. Φ_i obviously meets the infinity disappearing condition, that is

$$x = \infty, \quad \Phi_1 = 0, \quad y = -\infty, \quad \Phi_2 = 0, \tag{2.9}$$

For the two-material interface, the initial conditions should also be given. We only discuss harmonics with zero initial velocity of k:

$$\eta(x, 0) = \varepsilon \cos kx, \quad \eta_t(x, 0) = 0. \tag{2.10}$$

In general, for a variety of k harmonics, we can use the Fourier series expansion method, which can be solved by linear superposition method.

Since the equation is linear, the variable x appears only in the initial condition as $\cos kx$, so it is desirable to choose

$$\Phi_i(x, y, t) = \varphi_i(x, t) \cos kx, \quad \eta(x, t) = A(t) \cos kx. \tag{2.11}$$

We substitute this into (2.10) and get $A(0)=\varepsilon,\ A'(0)=0$. Also we insert them into Laplace equation, then obtain

$$\varphi_{iyy}-k^2\varphi_i=0. \tag{2.12}$$

Therefore the solution takes the following form: $\varphi_i=\mathrm{e}^{\pm ky}A_i(t)$. By the infinity condition (2.9), we know $\varphi_1=A_1(t)\mathrm{e}^{-ky},\ \varphi_2=A_2(t)\mathrm{e}^{ky}$. Substituting these results into the boundary condition $y=0$ in (2.7), we have

$$\rho_1(\ddot{A}_1(t)+gA_1(t))=\rho_2(\ddot{A}_2(t)+gA_2(t)),\quad \ddot{A}(t)=-k_1A_1=kA_2.$$

Differentiate the second equation of the above system with respect to t, then substitute it into the first equation of the above system, and eliminate $\ddot{A}_1(t)$ and $\ddot{A}_2(t)$. Finally, we obtain

$$\ddot{A}(t)-n_0^2A(t)=0,\quad n_0=\sqrt{\frac{\rho_1-\rho_2}{\rho_1+\rho_2}\cdot kg}. \tag{2.13}$$

Now we will discuss it in two situations.

(1) $\rho_1>\rho_2$, that is, the heavy fluid is located above the light fluid. At this time n_0 is purely real, $A(t)=a\mathrm{e}^{-n_0t}+b\mathrm{e}^{n_0t}$. Due to the existence of the e^{n_0t} formal solution, $A(t)$ grows exponentially over time, that is, the disturbance is unstable. This phenomenon is explained in Figure 2.2. When a fluid with a density of ρ_1 invades into a fluid with a density of ρ_2, the gravity of the portion of the fluid is $G\backsim V\rho_1g$.

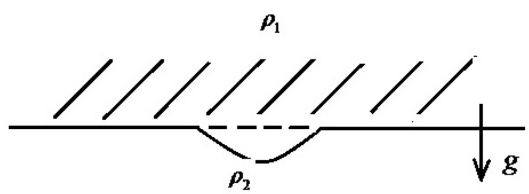

Figure 2.2

And its buoyancy is the weight of the invading fluid that is discharged in the same volume, that is, the buoyancy is $F\sim V\rho_2g$.

When $\rho_1>\rho_2$, the gravity is greater than the buoyancy, and this part of the invading fluid continues to sink, causing the disturbance on the interface to develop further, which leads to instability.

Taking into account the initial conditions (2.10), we have

$$y=\eta(x,t)=\varepsilon\cosh n_0t+\cos kx. \tag{2.14}$$

Here $\cosh n_0t=(\mathrm{e}^{n_0t}+\mathrm{e}^{-n_0t})/2$ is the hyperbolic cosine. If the initial condition (2.10) is changed to $\eta(x,0)=0,\ \eta_t(x,0)=\varepsilon\cos kx$, then the form of the interface is $y=\frac{\varepsilon}{n_0}\sinh nt\cos kx$.

In a more general case, it should be a linear superposition of two solutions.

After finding the expression of the interface shape $\eta(x,t)$, it is not difficult to get the form of $\varphi_i(t)$.

(2) $\rho_1 < \rho_2,\ n_0 = im_0$, where n_0 is the pure imaginary number. At this time

$$y = \eta(x,t) = \varepsilon \cos m_0 t + \cos kx. \tag{2.15}$$

Then the disturbance on the interface no longer develops, but it oscillates with time. The interface is stable corresponding to the gravity less than buoyancy in Figure 2.2, and the density ρ_1 of the fluid invading into the density ρ_2 of the fluid is retracted again, such that the interface disturbances are no longer considered.

Now we discuss the incompressible fluid with a density of ρ, a thickness of h under the gravitational field whose the direction is downward. In order for the fluid to be suspended and stationary, a pressure difference must be given to the lower surface. Since the ideal incompressible flow is initially at rest, it is curl-free.

As before, we can introduce the velocity potential. Let Φ^0 be the case corresponding to the undisturbed static, and φ be the additional velocity potential formed by the interface disturbance. The velocity potential is $\Phi = \Phi^{(0)} + \varphi$. In the case of ignoring high order quantities, the pressure can be expressed as $p = -\rho(\Phi_t^0 + \varphi_t + gyf(t))$. Firstly we take into account the zero-order situation.

On the upper surface $y = 0$, $p_1 = -\rho(\Phi_t^0 + f(t))$.

On the lower surface $y = -1$, $p_2 = -\rho(\Phi_t^{(0)} - gh + f(t))$.

Obviously, Φ_t^0 is only a function of t, then we obtain $p_2 - p_1 = \rho gh$.

Now discuss the disturbance on the interface. On the upper surface, we have $y = \eta_1(x,t)$; on the lower surface, there is $y = -h + \eta_2(x,t)$. Since the pressures of the upper and lower surfaces, p_1 and p_2, satisfy the zero-order quantity, the disturbance pressure is

$$y = 0, \quad \varphi_t + g\eta_1 = 0,$$
$$y = -h, \quad \varphi_t + g\eta_2 = 0.$$

2.2 Finite thickness double interface fluid

In practice, we often encounter complex situations of fluids with limited thickness.

In addition, the second condition of (2.5) can be linearized to

$$y = 0, \quad \eta_{1,t} = \varphi_y,$$
$$y = -h, \quad \eta_{2,t} = \varphi_y.$$

The perturbation of the Fourier component with $\cos kx$ is still discussed, so that φ satisfies the Laplace equation, by choosing $\varphi = (A\mathrm{e}^{ky} + B\mathrm{e}^{-ky})\mathrm{e}^{nt}\cos kx$, $\eta_1 = a_1\mathrm{e}^{nt}\cos kx$ and $\eta_2 = a_2\mathrm{e}^{nt}\cos kx$. Substituting these into the boundary condition of $y = 0$, we get

$$n(A+B)+ga_1=0, \quad na_1=(A-B)k.$$

Substituting into the condition of $y=-h$, we have

$$n(A\mathrm{e}^{-kh}+B\mathrm{e}^{kh})+ga_2=0, \quad na_2=k(A\mathrm{e}^{-kh}-B\mathrm{e}^{kh}).$$

Eliminating a_1 and a_2 from these four equations, we get

$$A(n^2+kg)+B(n^2-kg)=0, \quad A(n^2+kg)\mathrm{e}^{-kh}+B(n^2-kg)\mathrm{e}^{kh}=0.$$

Thus, the conditions for the non-trivial solutions of A and B are $n^2=\mp kg$.

For simplicity, we take $m_0=\sqrt{kg}$. Then the two real roots of n are $n=\pm m_0$. Obviously for $A=0$, $\varphi=B\mathrm{e}^{-ky}\mathrm{e}^{\pm m_0 t}\cos kx$ is the solution of the lower surface problem. For $B=0$, $\varphi=A\mathrm{e}^{ky}\mathrm{e}^{\pm i m_0 t}\cos kx$ is the solution of the upper surface problem. When $t=0$, the fluid should be at rest, so the solution in the general form can be written as $\varphi=[\overline{A}\sin(m_0t)\mathrm{e}^{ky}+B\mathrm{sh}(m_0t)\mathrm{e}^{-ky}]\cos kx$.

Obviously the first term in square brackets represents a solution that vibrates over time, while the second term represents an unstable solution. Correspondingly, the two surface disturbances should also have the following form of solution:

$$\eta_1=[A_1\cos m_0t+B_1\mathrm{ch}\, m_0t]\cos kx, \quad \eta_2=[A_2\cos m_0t+B_2\mathrm{ch}\, m_0t]\cos kx.$$

Substituting these into the kinematic conditions of $y=0,\ y=-h$, we can get

$$m_0[-A_1\sin m_0t+B_1\mathrm{sh}\, m_0t]=k[\overline{A}\sin m_0t-\overline{B}\mathrm{sh}\, m_0t],$$
$$m_0[-A_2\sin m_0t+B_2\mathrm{sh}\, m_0t]=k[\overline{A}\mathrm{e}^{-kh}\sin m_0t-\overline{B}\mathrm{e}^{kh}\mathrm{sh}\, m_0t].$$

Therefore,

$$-\overline{A}=\frac{m_0A_1}{k}=\frac{m_0A_2\mathrm{e}^{kh}}{k}, \quad -\overline{B}=\frac{m_0B_1}{k}=\frac{m_0B_2\mathrm{e}^{-kh}}{k},$$

where $A_2=A_1\mathrm{e}^{-kh}$, $B_2=B_1\mathrm{e}^{kh}$. Assuming that the upper and lower surface disturbances at the initial moment $t=0$ are $\eta_1=\varepsilon_1\cos kx$ and $\eta_2=\varepsilon_2\cos kx$. Obviously we have $A_1+B_1=\varepsilon_1$ and $A_2+B_2=\varepsilon_2$, which can be written as $A_1\mathrm{e}^{-kh}+B_1\mathrm{e}^{kh}=\varepsilon_2$. Thus, we have

$$A_1=\frac{\varepsilon_1-\varepsilon_2\mathrm{e}^{-kh}}{1-\mathrm{e}^{-2kh}}, \quad B_1=\frac{\varepsilon_2-\varepsilon_1}{1-\mathrm{e}^{-2kh}}\mathrm{e}^{-kh}, \quad A_2=\frac{\varepsilon_1-\varepsilon_2\mathrm{e}^{-kh}}{1-\mathrm{e}^{-2kh}}\mathrm{e}^{-kh}, \quad B_2=\frac{\varepsilon_2-\varepsilon_1\mathrm{e}^{-kh}}{1-\mathrm{e}^{-2kh}},$$

and

$$\begin{cases} \eta_1(x,t)=\dfrac{1}{1-\mathrm{e}^{-2k\eta}}\left[(\varepsilon_1-\varepsilon_2\mathrm{e}^{-kh})\cos m_0t+(\varepsilon_2-\varepsilon_1\mathrm{e}^{-kh})\mathrm{e}^{-kh}\cosh m_0t\right]\cos kx, \\ \eta_2(x,t)=\dfrac{1}{1-\mathrm{e}^{-2kh}}\left[(\varepsilon_1-\varepsilon_2\mathrm{e}^{-kh})\mathrm{e}^{-kh}\cos m_0t+(\varepsilon_2-\varepsilon_1\mathrm{e}^{-kh})\cosh m_0t\right]\cos kx. \end{cases} \tag{2.16}$$

For $\varepsilon_2 = 0$ and $\varepsilon_1 = 0$, the expression of (2.16) is very interesting, so it can be seen as the correlation and perturbation between the upper and lower surfaces.

2.3 Fluid interface movement of finite thickness two-sided vacuum

As is shown in Figure 2.3, there is a finite thickness of d incompressible fluid with a density of ρ, where outside the horizontal infinite upper and lower surfaces is vacuum, and the gravity field force is $\overrightarrow{g} = g\overrightarrow{e_y}$.

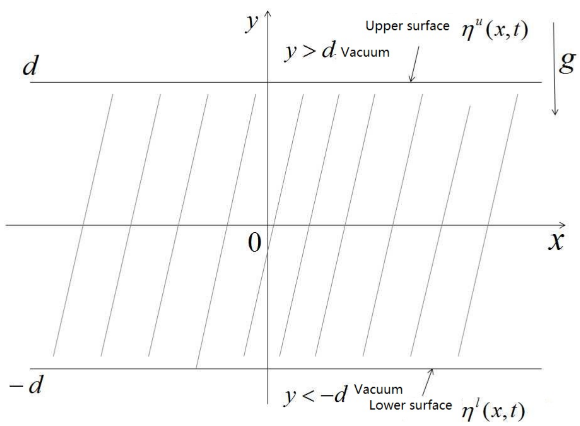

Figure 2.3

Set

$$F^u(x, y, t) = y - \eta^u(x, t), \tag{2.17}$$

$$F^l(x, y, t) = y + d - \eta^l(x, t), \tag{2.18}$$

which denote the equations of motion for the upper and lower interfaces respectively, where $\eta^u(x, t)$ and $\eta^l(x, t)$ respectively are perturbations of the upper and lower interfaces. Again, letting $\Phi^0(t)$ be the initial undisturbed velocity potential, the velocity potential of the disturbing fluid is $\Phi(x, y, t) = \Phi^0(t) + \varphi(x, y, t)$. Here φ is the additional velocity potential caused by the disturbance interface with $\nabla^2\varphi = 0$. According to the Bernoulli equation, the pressures on the upper and lower interfaces are

$$p^u = -\rho\Big[\frac{\partial\Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2 + g\eta^u + f(t)\Big], \tag{2.19}$$

$$p^l = -\rho\Big[\frac{\partial\Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2 - gd + g\eta^l + f(t)\Big], \tag{2.20}$$

where $f(t)$ is a function. For the undisturbed situations,

$$p^{u(0)} = -\rho\Big[\frac{\partial\Phi^0}{\partial t} + f(t)\Big], \quad p^{l(0)} = -\rho\Big[\frac{\partial\Phi^0}{\partial t} + f(t)\Big], \tag{2.21}$$

which satisfy the pressure balance condition $p^{l(0)} - p^{u(0)} = \rho g d$. Therefore the disturbance pressure is zero on both interfaces, that is,

$$\rho\Big[\frac{\partial\varphi}{\partial t}+\frac{1}{2}(\nabla\varphi)^2+g\eta^u\Big]=0,\quad y=\eta^u, \tag{2.22}$$
$$\rho\Big[\frac{\partial\varphi}{\partial t}+\frac{1}{2}(\nabla\varphi)^2+g\eta^l\Big]=0,\quad y=\eta^l. \tag{2.23}$$

As $[-\frac{\partial F}{\partial t}/|\nabla F|]$ is known as interface normal velocity and $\overrightarrow{n}=\nabla F/|\nabla F|$ is the normal vector, we have

$$-\frac{\frac{\partial F^u}{\partial t}}{|\nabla F^u|}=\nabla\Phi\cdot\frac{\nabla F^u}{|\nabla F^u|},\quad y=\eta^u, \tag{2.24}$$
$$-\frac{\frac{\partial F^l}{\partial t}}{|\nabla F^l|}=\nabla\Phi\cdot\frac{\nabla F^l}{|\nabla F^l|},\quad y=\eta^l. \tag{2.25}$$

Substituting these into equations (2.17) and (2.18), we get

$$\frac{\partial\eta^u}{\partial t}=-\frac{\partial\varphi}{\partial x}\frac{\partial\eta^u}{\partial x}+\frac{\partial\varphi}{\partial y},\quad y=\eta^u, \tag{2.26}$$
$$\frac{\partial\eta^l}{\partial t}=-\frac{\partial\varphi}{\partial x}\frac{\partial\eta^l}{\partial x}+\frac{\partial\varphi}{\partial y},\quad y=\eta^l. \tag{2.27}$$

According to (2.22), (2.23), (2.26), (2.27), we expand η^u,η^l,φ as the power of the small parameter ε and obtain

$$\begin{aligned}
\eta^u(x,t)&=\varepsilon\eta^u_{1,1}(t)\cos kx+\varepsilon^2[\eta^u_{2,0}(t)+\eta^u_{2,2}(t)\cos 2kx]+\cdots,\\
\eta^l(x,t)&=\varepsilon\eta^l_{1,1}(t)\cos kx+\varepsilon^2[\eta^l_{2,0}(t)+\eta^l_{2,2}(t)\cos 2kx]+\cdots,\\
\varphi(x,y,t)&=\varepsilon[\mathrm{e}^{ky}A_{1,1}(t)+\mathrm{e}^{-ky}B_{1,1}(t)]\cos kx\\
&\quad+\varepsilon^2\{A_{2,0}(t)+[\mathrm{e}^{2ky}A_{2,2}(t)+\mathrm{e}^{-2ky}B_{2,2}(t)]\cos 2kx\}+\cdots.
\end{aligned}$$

In the case of the first-order approximation $o(\varepsilon)$, we can obtain a second-order coupled ordinary differential equation system

$$\frac{\mathrm{d}^2\eta^u_{1,1}}{\mathrm{d}t^2}-\frac{2\mathrm{e}^{-\xi}}{1+\mathrm{e}^{-2\xi}}\frac{\mathrm{d}^2\eta^l_{1,1}}{\mathrm{d}t^2}+\gamma^2\frac{1-\mathrm{e}^{-2\xi}}{1+\mathrm{e}^{2\xi}}\eta^u_{1,1}=0, \tag{2.28}$$
$$\frac{\mathrm{d}^2\eta^l_{1,1}}{\mathrm{d}t^2}-\frac{2\mathrm{e}^{-\xi}}{1+\mathrm{e}^{-2\xi}}\frac{\mathrm{d}^2\eta^u_{1,1}}{\mathrm{d}t^2}-\gamma^2\frac{1-\mathrm{e}^{-2\xi}}{1+\mathrm{e}^{2\xi}}\eta^l_{1,1}=0, \tag{2.29}$$

where $\gamma^2=gk$ is the square of the linear growth rate of classical $RT1(A_T=1)$, $\xi=kd$.

The initial conditions are

$$\eta^u_{1,1}|_{t=0}=a,\quad \frac{\mathrm{d}\eta^u_{1,1}}{\mathrm{d}t}\Big|_{t=0}=0,\quad \eta^l_{1,1}|_{t=0}=b,\quad \frac{\mathrm{d}\eta^l_{1,1}}{\mathrm{d}t}\Big|_{t=0}=0. \tag{2.30}$$

From these, we get

$$\eta_{1,1}^u(t) = \frac{1}{1-\mathrm{e}^{-2\xi}}[\overline{a}\cos\gamma t + \mathrm{e}^{-\xi}\overline{b}\cosh\gamma t], \tag{2.31}$$

$$\eta_{1,1}^l(t) = \frac{1}{1-\mathrm{e}^{-2\xi}}[\mathrm{e}^{-\xi}\overline{a}\cos\gamma t + \overline{b}\cosh\gamma t], \tag{2.32}$$

where $\overline{a} = a - b\mathrm{e}^{-\xi}$, $\overline{b} = b - a\mathrm{e}^{-\xi}$.

In the second order of $o(\varepsilon^2)$, the coupled ODE is

$$\begin{aligned}&\frac{\mathrm{d}^2\eta_{2,2}^u}{\mathrm{d}t^2} - \frac{2\mathrm{e}^{-2\xi}}{1+\mathrm{e}^{-4\xi}}\frac{\mathrm{d}^2\eta_{2,2}^l}{\mathrm{d}t^2} + 2\gamma^2\frac{1-\mathrm{e}^{-4\xi}}{1+\mathrm{e}^{-4\xi}}\eta_{2,2}^u + \gamma^2 k\Big[\frac{1}{2}\overline{a}^2 c_1\cosh 2T\\&-\frac{1}{2}\mathrm{e}^{-2\xi}c_1\overline{b}^2\cos 2T - \frac{2}{\mathrm{e}^{2\xi}+\mathrm{e}^{-2\xi}}c_{ab}\cos T\cosh T\\&+2\mathrm{e}^{-2\xi}c_1c_{ab}\sin T\sinh T + \frac{\mathrm{e}^{\xi}+\mathrm{e}^{2\xi}}{1+\mathrm{e}^{4\xi}}a\overline{b}\Big] = 0,\end{aligned} \tag{2.33}$$

$$\begin{aligned}&\frac{\mathrm{d}^2\eta_{2,2}^l}{\mathrm{d}t^2} - \frac{2\mathrm{e}^{-2\xi}}{1+\mathrm{e}^{-4\xi}}\frac{\mathrm{d}^2\eta_{2,2}^u}{\mathrm{d}t^2} - 2\gamma^2\frac{1-\mathrm{e}^{-4\xi}}{1+\mathrm{e}^{4\xi}}\eta_{2,2}^l + \gamma^2 k\Big[\frac{1}{2}\overline{b}^2 c_1\cosh 2T\\&-\frac{1}{2}\mathrm{e}^{-2\xi}c_1\overline{a}^2\cos 2T - \frac{2}{\mathrm{e}^{2\xi}+\mathrm{e}^{-2\xi}}c_{ab}\cos T\cosh T\\&-2\mathrm{e}^{-2\xi}c_1c_{ab}\sin T\sinh T + \frac{\mathrm{e}^{\xi}+\mathrm{e}^{2\xi}}{1+\mathrm{e}^{4\xi}}b\overline{a}\Big] = 0,\end{aligned} \tag{2.34}$$

where $\overline{a} = a - b(\mathrm{e}^{-\xi}+\mathrm{e}^{\xi})/2$, $\overline{b} = b - a(\mathrm{e}^{-\xi}+\mathrm{e}^{\xi})/2$, $c_1 = (1+\mathrm{e}^{-2\xi})/(1-\mathrm{e}^{-2\xi}+\mathrm{e}^{-4\xi}-\mathrm{e}^{-6\xi})$, $c_{ab} = a^2+b^2-ab(\mathrm{e}^{-\xi}+\mathrm{e}^{\xi})$, $T=\gamma t$ and the initial conditions are

$$\begin{aligned}&\eta_{2,2}^u(t=0)=0, \quad \frac{\mathrm{d}\eta_{2,2}^u}{\mathrm{d}t}(t=0)=0,\\&\eta_{2,2}^l(t=0)=0, \quad \frac{\mathrm{d}\eta_{2,2}^l}{\mathrm{d}t}(t=0)=0.\end{aligned} \tag{2.35}$$

From these, we deduce

$$\begin{aligned}\eta_{2,2}^u(t) = &\frac{k}{2}\Big(\frac{1}{1-\mathrm{e}^{-2\xi}}\Big)^2\Big[-\mathrm{e}^{-2\xi}\overline{b}^2\cosh^2 T + \overline{a}^2\cos^2 T - \frac{\mathrm{e}^{-2\xi}}{1+\mathrm{e}^{-2\xi}}D_1\overline{b}\cosh\sqrt{2}T\\&-\frac{1}{1+\mathrm{e}^{-2\xi}}D_2\overline{a}\cos\sqrt{2}T - 2\mathrm{e}^{-\xi}\frac{1-\mathrm{e}^{-2\xi}}{1+\mathrm{e}^{-2\xi}}\overline{a}\overline{b}\cos T\cosh T\Big],\end{aligned} \tag{2.36}$$

$$\begin{aligned}\eta_{2,2}^l(t) = &\frac{k}{2}\Big(\frac{1}{1-\mathrm{e}^{-2\xi}}\Big)^2\Big[-\overline{b}^2\cosh^2 T + \mathrm{e}^{-2\xi}\overline{a}^2\cos^2 T - \frac{1}{1+\mathrm{e}^{-2\xi}}D_1\overline{b}\cosh\sqrt{2}T\\&-\frac{\mathrm{e}^{-2\xi}}{1+\mathrm{e}^{-2\xi}}D_2\overline{a}\cos\sqrt{2}T + 2\mathrm{e}^{-\xi}\frac{1-\mathrm{e}^{-2\xi}}{1+\mathrm{e}^{-2\xi}}\overline{a}\overline{b}\cos T\cosh T\Big],\end{aligned} \tag{2.37}$$

which are shown in Figure 2.4.

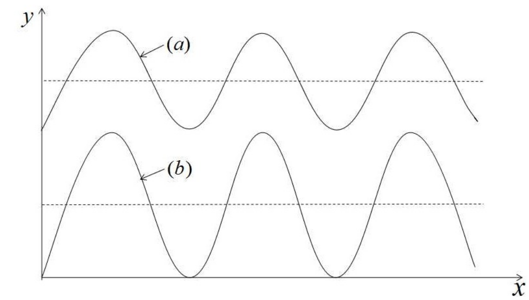

Figure 2.4: (a) $kd=0.2,\ \gamma t=4.2,\ \cdots$, (b) $kd=0.8,\ \gamma t=4.9$

2.4 Variable density fluid layer

Here we don't discuss a two-layer fluid with a density discontinuity, but a single-layer fluid with a density distribution in the y direction. This distribution is also unstable under certain conditions. The fluid is heated from the bottom and has different densities at different temperatures. Then it leads to convection. Without disturbance, the fluid is at rest. At this time

$$\frac{\mathrm{d}p^{(0)}}{\mathrm{d}y}=-\rho^{(0)}(y)g.$$

Therefore we get

$$p^{(0)}=-\int\rho^{(0)}(y)g\mathrm{d}y+f(t).$$

In case of disturbance

$$p=p^{(0)}+p',\quad \rho=\rho^{(0)}+\rho'.$$

And u,v are first-order small quantities. The continuity equation can be written as

$$\frac{\partial\rho'}{\partial t}+\rho^{(0)}\Big(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\Big)+v\frac{\mathrm{d}\rho^{(0)}}{\mathrm{d}y}=0.$$

We take

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,$$

then,

$$\frac{\partial\rho'}{\partial t}=-v\frac{\mathrm{d}\rho^{(0)}}{\mathrm{d}y}.$$

In the momentum equation, for the disturbance term, we have

$$\rho^{(0)}\frac{\partial u}{\partial t}+\frac{\partial p'}{\partial x}=0,\quad \rho^{(0)}\frac{\partial v}{\partial t}+\frac{\partial p'}{\partial y}=-\rho' g.$$

Assume that the disturbance has the form e^{ikx+nt}, that is

$$p'=\overline{p}(y)\mathrm{e}^{nt+ikx},\ \ \rho'=\overline{\rho}(y)\mathrm{e}^{nt+ikx},\ \ u'=\overline{u}(y)\mathrm{e}^{nt+ikx},\ \ v'=\overline{v}(y)\mathrm{e}^{nt+ikx}.$$

Substituting these into the above equation, we get

$$ik\overline{u} = -\frac{\mathrm{d}\overline{v}}{\mathrm{d}y}, \quad n\overline{\rho} = -\overline{v}\frac{\mathrm{d}\rho^{(0)}}{\mathrm{d}y}, \quad \rho^{(0)}n\overline{u} + ik\overline{p} = 0, \quad \rho^{(0)}n\overline{v} + \frac{\mathrm{d}\overline{p}}{\mathrm{d}y} = -g\overline{\rho}.$$

Eliminating $\overline{u}$ from the first and third equations, we deduce

$$\rho^{(0)}n\frac{\mathrm{d}\overline{v}}{\mathrm{d}y} + k^2\overline{p} = 0.$$

And by the second and fourth equations, there is

$$\rho^{(0)}n\overline{v} + \frac{\mathrm{d}\overline{p}}{\mathrm{d}y} = \frac{\overline{v}}{n}g\frac{\mathrm{d}\rho^{(0)}}{\mathrm{d}y}.$$

Eliminating $\overline{p}$ from the last two equations, we have

$$\frac{\mathrm{d}}{\mathrm{d}y}\Big(\rho^{(0)}\frac{\mathrm{d}\overline{v}}{\mathrm{d}y}\Big) - k^2\rho^0\overline{v} + \frac{k^2g}{n^2}\frac{\mathrm{d}\rho^{(0)}}{\mathrm{d}y}\overline{v} = 0. \tag{2.38}$$

If you do not consider the density distribution in the y direction, $\frac{\mathrm{d}\rho^{(0)}}{\mathrm{d}y} = 0$, then

$$\frac{\mathrm{d}^2\overline{v}(y)}{\mathrm{d}y^2} - k^2\overline{v}(y) = 0.$$

That is, $\overline{v} = a\mathrm{e}^{\pm ky}$. Now we discuss the case of $\rho^{(0)} = \rho^{(0)}(y)$. For simplicity, we only study the case:

$$\rho^{(0)} = \rho_0\mathrm{e}^{\beta y}, \quad \beta = \text{const}.$$

Substituting these into (2.38), we have

$$\frac{\mathrm{d}^2\overline{v}}{\mathrm{d}y^2} + \beta\frac{\mathrm{d}\overline{v}}{\mathrm{d}y} - k^2\Big(1 - \frac{g\beta}{n^2}\Big)\overline{v} = 0. \tag{2.39}$$

Taking the form solution of $\overline{v} \sim \mathrm{e}^{qy}$, there is

$$q^2 + \beta q - k^2\Big(1 - \frac{g\beta}{n^2}\Big) = 0.$$

Therefore, we have

$$\overline{v}(y) = A_1\mathrm{e}^{q_1y} + A_2\mathrm{e}^{q_2y},$$

Here q_1, q_2 is the two roots for the above equation, that is

$$q_1 = \frac{1}{2}\Bigg\{-\beta + \sqrt{\beta^2 + 4k^2\Big(1 - \frac{g\beta}{n^2}\Big)}\Bigg\}, \quad q_2 = \frac{1}{2}\Bigg\{-\beta - \sqrt{\beta^2 + 4k^2\Big(1 - \frac{g\beta}{n^2}\Big)}\Bigg\}. \tag{2.40}$$

Assuming that the layer of fluid is between the two solid walls of $y = -h$ and $y = 0$, by $y = 0$, $\overline{v}(y) = 0$, we get $A_2 - A_1 = A$, and $\overline{v}(y) = A(\mathrm{e}^{q_1y} - \mathrm{e}^{q_2y})$. Due to $y = -h$, $\overline{v}(y) = 0$, we have $\mathrm{e}^{(q_1-q_2)h} = 1$, which holds only when

$$q_1 - q_2 = \frac{2m\pi i}{h}. \tag{2.41}$$

Thus,

$$\overline{v}(y) = A\mathrm{e}^{\frac{1}{2}(q_1+q_2)y}\{\mathrm{e}^{\frac{1}{2}(q_1-q_2)y} - \mathrm{e}^{-\frac{1}{2}(q_1-q_2)y}\}.$$

In accordance with (2.40), (2.41), we have

$$\overline{v} = B\mathrm{e}^{-\frac{1}{2}\beta y}\sin\frac{m\pi y}{h} \tag{2.42}$$

and

$$n = \frac{\sqrt{g\beta}}{\sqrt{1+\big[\frac{\beta^2}{4}+\frac{m^2\pi^2}{h^2}\big]k^{-2}}}. \tag{2.43}$$

It can be seen from the above equation that $\beta > 0$, that is, as the density gradient of the fluid is the same direction as g, the above density distribution of the fluid layer is unstable. For short waves,

$$n \sim \sqrt{g\beta}.$$

That is, the index n is proportional to $\beta^{\frac{1}{2}}$, the greater the density gradient, the more unstable the instability.

2.5 The Taylor instability of viscous fluids

The two-layer semi-infinite fluids occupy the $y > 0$ and $y < 0$ planes, respectively, with densities of ρ_1 and ρ_2. The gravitational acceleration is perpendicular to the interface, along the negative direction of y. In the case of incompressibility, the continuous equation can be written as

$$u_x + v_y = 0.$$

The momentum equation is the following NS equation

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u + \frac{1}{\rho}\nabla p = \frac{\mu}{\rho}\Delta u + g.$$

On the material interface $F(\vec{r}, t) = 0$, the interface normal velocity is continuous with

$$-\frac{\partial F}{\partial t} = (u_1\cdot\nabla)F = (u_2\cdot\nabla)F,$$

And the tangential velocity is continuous, the stress equation is

$$n_j\sigma_{1jk} = n_j\sigma_{2jk},$$

where σ_{ijk} $(i = 1, 2)$ is the viscous stress tensor of the two fluids with

$$\sigma_{ijk} = -p_i\delta_{jk} + \mu_i\Big(\frac{\partial u_{ij}}{\partial x_i} + \frac{\partial u_{ik}}{\partial x_j}\Big), \quad i = 1, 2,$$

p_i is the fluid pressure, and $i = 1, 2$ means two different fluids.

Set the form of the interface to be $F(\vec{r},t) = y - \eta(x,t) = 0$, where $\vec{\eta}(x,t)$ is a small amount, then $\vec{n} = -\eta_x\vec{i} + \vec{j}$. And the corresponding u, p are first-order small quantities, then the momentum equation can be linearized to

$$u_{it} + \frac{p_{ix}}{\rho_i} = \frac{\mu_i}{\rho_i}(u_{ixx} + u_{iyy}), \quad v_{it} + \frac{p_{iy}}{\rho_i} = \frac{\mu_i}{\rho_i}(v_{ixx} + v_{iyy}) - g, \quad i = 1, 2.$$

For viscous fluids, the flow is not no-curl, since the potential function is insufficient to satisfy the momentum equation. We choose

$$u_i = \Phi_{ix} + \Psi_{iy}, \quad v_i = \Phi_{iy} - \Psi_{ix}, \quad i = 1, 2.$$

Substituting this into the momentum equation, we have

$$\Big(\Phi_{it} + \frac{P_i}{\rho_i}\Big)_x = \frac{\mu_i}{\rho_i}(\Psi_{ixx} + \Psi_{iyy})_y - \Psi_{ity}, \quad \Big(\Phi_{it} + \frac{P_i}{\rho_i}\Big)_y = \frac{\mu_i}{\rho_i}(\Psi_{ixx} + \Psi_{iyy})_x - \Psi_{itx} - g.$$

If we set $\Psi_{it} = \frac{\mu_i}{\rho_i}(\Psi_{ixx} + \Psi_{iyy})$, then

$$\Phi_{it} + \frac{P_i}{\rho_i} + gy = f(t), \quad p_i = -\rho_i(\Phi_{it} + gy).$$

According to the existing vector expression of $\vec{n}$, the equation of motion and stress conditions on the interface can be written as

$$y = 0, \quad u_1 = u_2, \quad \eta_t = v_1 = v_2, \quad \sigma_{1xy} = \sigma_{2xy},$$
$$\sigma_{1xx} - \sigma_{2yy} = \Sigma\Big(\frac{1}{R_1} + \frac{1}{R_2}\Big) = -\Sigma\eta_{xx}.$$

Here, the surface tension is considered in the normal pressure difference, Σ is the surface tension coefficient, and the corresponding stress is

$$\sigma_{ixy} = \mu_i\Big[\frac{\partial u_i}{\partial y} + \frac{\partial v_i}{\partial x}\Big], \quad \sigma_{iyy} = -p_i + 2\mu\frac{\partial v_i}{\partial \eta}.$$

Representing u_i, v_i as a function expression for Φ_i and Ψ_i, substituting them into the above expression and converting the condition on the interface $y = 0$ to

$$\begin{cases} \Phi_{1x} + \Psi_{1y} = \Phi_{2x} + \Psi_{2y}, \\ \eta_t = \Phi_{1y} - \Psi_{1x} = \Phi_{2y} - \Psi_{2x}, \end{cases} \tag{2.44}$$

we obtain

$$\mu_1[2\Phi_{1xy} + \Psi_{1yy} - \Psi_{1xx}] = \mu_2[2\Phi_{2xy} + \Psi_{2yy} - \Psi_{2xx}],$$
$$\rho_1(\Phi_{1t} + g\eta) + 2\mu_1(\Phi_{1yy} - \Psi_{1xy}) + \Sigma\eta_{xx} = \rho_2(\Phi_{2t} + y\eta) + 2\mu_2(\Phi_{2yy} - \Psi_{2xy}). \tag{2.45}$$

Assume that the initial perturbation of the interface is $\eta(x,0) = \varepsilon\cos kx$. In order for Φ_i to satisfy the conditions of the Laplace equation and $x = \pm\infty$, we take

$$\eta(x,t) = a\mathrm{e}^{nt}\cos kx, \quad \Phi_1 = A\mathrm{e}^{ht-ky}\cos kx, \quad \Phi_2 = C\mathrm{e}^{nt+ky}\cos kx.$$

Set

$$\Psi_1 = B\mathrm{e}^{-m_1 y+nt}\sin kx, \quad \Psi_2 = D\mathrm{e}^{m_2 y+nt}\sin kx,$$

Substituting these into (2.43) and considering the disappearance condition at $y = \pm\infty$, we deduce

$$m_i = \sqrt{k^2 + \frac{\rho_i n}{\mu_i}}.$$

Substituting the above Φ_i and Ψ_i expressions into the inner boundary conditions (2.44) and (2.45), eliminating a, we get

$$\begin{gathered} A + B + C + D = 0, \\ kA + m_1 B - kC + m_2 D = 0, \\ 2\mu_1 k^2 A + \mu_1(m^2 + k^2)B + 2\mu_2 k^2 C - \mu_2(m_2^2 + k^2)D = 0, \\ \Big(\frac{a}{h} - n\rho_1 - 2\mu_1 k^2\Big)A + \Big(\frac{a}{n} - 2\mu_1 m_1 k\Big)B + (\rho_2 n + 2\mu_2 k^2)C - 2\mu_2 m_2 k D = 0. \end{gathered}$$

Here $a = (\rho_1 - \rho_2)kg - k^3\Sigma$.

The non-trivial solution of the above homogeneous equations of A, B, C and D exists under the condition that the corresponding coefficient determinant is zero, that is

$$\begin{vmatrix} 1 & 1 & 1 & -1 \\ k & m_1 & -k & m_2 \\ 2\mu_1 & \mu_1(k^2 + m_1^2) & 2\mu_2 k^2 & -\mu_2(k^2 + m_2^2) \\ \frac{a}{n} - n\rho_1 - 2\mu_1 k^2 & \frac{a}{n} - 2\mu_1 m_1 k & n\rho_2 + 2\mu_2 k^2 & -2\mu_2 m_2 k \end{vmatrix} = 0.$$

Expanding this determinant and substituting m into this, we get

$$[-a + (\rho_1 + \rho_2)n^2]\Big\{\frac{1}{\mu_1 k + \sqrt{\mu_2^2 + \mu_2\rho_2 n}} + \frac{1}{\mu_2 k + \sqrt{\mu_1^2 + \mu_1\rho_1 n}}\Big\} + 4nk = 0. \quad (2.46)$$

When the viscosity μ_i is large, n becomes smaller. We ignore the term with n in the root of (2.46), that is take $\mu_i k^2 \gg \rho_i n$, or approximately choose $m_i \backsimeq k$. Thus we get

$$[-a + (\rho_1 + \rho_2)n^2]\frac{2}{k(\mu_1 + \mu_2)} + 4nk = 0.$$

Meanwhile we choose $\nu = (\mu_1 + \mu_2)/(\rho_1 + \rho_2)$, therefore

$$n = -\nu k^2 \pm \sqrt{\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}kg + \nu^2 k^4 - \frac{k^4\Sigma}{\rho_1\rho_2}}. \quad (2.47)$$

We focus on the unstable root where the coefficient before the square root is positive, so that in the case of no viscosity, the index n is n_0

$$n_0 = \sqrt{\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} kg - \frac{k^3 \Sigma}{\rho_1 \rho_2}}. \tag{2.48}$$

We discuss it two situations.

(1) $n_0 \gg \nu k^2$, that is, when k or viscosity is small, by (2.47) we obtain

$$n = n_0 \Big[1 - \frac{\nu k^3}{n_0} + \frac{1}{2} \Big(\frac{\nu k^2}{n_0} \Big)^2 - \cdots \Big]. \tag{2.49}$$

From these we can see the viscosity reduces the RT instability. The larger k is, the more significant it is.

(2) $n_0 \ll \nu k^2$, by (2.47) we get

$$n = n_0 \frac{n_0}{2\nu k^2}. \tag{2.50}$$

That is, n quickly decreases with the increase of k, especially as $k \to k_c$. By (2.48) we can obtain

$$n_0 = \sqrt{\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} kg \Big(1 - \frac{k^2}{k_c^2} \Big)}.$$

This is the case where n tends to zero near k_c. When $\Sigma = 0$, regardless of surface tension,

$$n \sim \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} \frac{g}{2\nu k}.$$

That is, when k is large enough, n asymptotically approaches the k axis in the form of a curve.

2.6 Ablation instability

We consider a simplified model, discuss the ideal incompressible fluid, and assume that the effect of heat conduction is mainly the motion that derives the phase change. It has a certain velocity or acceleration, which is ignored in the equation. Let the section be at an average speed of U_0 moves forward in the direction of y with an acceleration of g, the densities before and after the cross section are ρ_1 and ρ_2 respectively, taking the coordinate system moving along the section, here the speed of the "1" zone in the coordinate system is

$$V_1 = -U_0, \tag{2.51}$$

along the opposite direction of y.

Conservation of mass momentum in a coordinate system with a constant cross-section can be written as

$$\begin{aligned} &\rho_1 q_1 \cdot n = \rho_2 q_2 \cdot n, \\ &p_1 + \rho_2 (q \cdot n)^2 = p_2 + (q_2 \cdot n)^2, \\ &q_1 \cdot \tau = q_2 \cdot \tau, \end{aligned} \tag{2.52}$$

here n and τ represent the normal and tangential directions of the section, respectively. ρ_i, p_i $(i=1,2)$ are respectively the densities and pressures on both sides, q_i $(i=1,2)$ denote the velocities of the section on both sides that are moving in motion. Assume that the section is the following form:

$$y=\eta(x,t),$$

where η is a small amount, then

$$\vec{n}=-\eta_x\vec{i}+\vec{j},\quad \tau=\vec{i}+\eta_x\vec{j},\quad D_n=\eta_t.$$

Here D_n is the normal velocity of the section. In this way, the pressure and speed can be expanded to

$$p_i=p_i^{(0)}+p_i^{(1)}+\cdots,\quad u_i=V_i\vec{j}+u_i^{(1)}+\cdots. \tag{2.53}$$

Here $p_i^{(0)}$ and $V_i\vec{j}$ are the zero-order quantities corresponding to the undisturbed case, and the second term is the first-order small quantity. The previously selected coordinate system is just the coordinate system that does not move the no section between the no disturbances, rather than that corresponds to the point on the section between the disturbances. In order to apply the conservation condition on the section, the following local coordinate transformation must be taken

$$q_i=u_i-D_nn=u_i-\eta_tj. \tag{2.54}$$

Substituting the relevant quantities into (2.52) respectively we can obtain the relationship between the following zero-order and first-order quantities on both sides of the discontinuity:

The zero order quantity

$$\rho_1V_1=\rho_2V_2=m,\quad p_1^{(0)}+\rho_1V_1^2=p_2^{(1)}+\rho_2V_2^2, \tag{2.55}$$

and the first order quantity

$$\begin{aligned}&\rho_1(v_1^{(1)}-\eta_t)=\rho_2(v_2^{(1)}-\eta_t),\\&p_1^{(1)}+2m(v_1^{(1)}-\eta_t)=p_2^{(1)}+2m(v_2^{(1)}-\eta_t),\\&u_1^{(1)}+V_1\eta_x=u_2^{(1)}+V_2\eta_x.\end{aligned} \tag{2.56}$$

Next we solve the first-order quantity. The front of the ablation section is originally static and therefore non-curl. However, the rear of the section, that is, the “2” area may be non-rotating, but may also be rotated. We discuss two cases below.

Case 1 There is no swirling flow after the cross section, since it is non-rotating in both areas, we can choose

$$u_i^{(1)}=\nabla\Phi_i,\quad i=1,2. \tag{2.57}$$

Under incompressible conditions, we have

$$\Delta\Phi_i = 0, \quad i = 1, 2. \tag{2.58}$$

Integrating the momentum equation and taking the first-order small quantity, the following pressure disturbance expression can be obtained:

$$p_i^{(1)} = -\rho_i\{\Phi_{it} + V_i\Phi_{iy} + gy + f_i(t)\}. \tag{2.59}$$

Substituting this into (2.56), we can get the condition of the section between $y = \eta(x,t)$. Since the zero-order quantity is uniform, the value of the first-order quantity on $y = \eta(x,t)$ can be Taylor expanded. Ignoring the high order small quantity, we deduce the condition on the $y = 0$ fixed boundary

$$\begin{aligned} &\rho_1(\Phi_{1y} - \eta_t) = \rho_2(\Phi_{2y} - \eta_t), \\ &\rho_1\{\Phi_{1t} - V_1\Phi_{1y} + g\eta\} = \rho_2\{\Phi_{2t} - V_2\Phi_{2y} + g\eta\}, \\ &\Phi_{1x} + V_1\eta_x = \Phi_{2x} + V_2\eta_x. \end{aligned} \tag{2.60}$$

For simplicity, we discuss the case where both layers are semi-infinite fluids, by assuming that a fluid with a density of ρ_1 occupies a semi-infinite plane of $y > 0$, and a fluid with a density of ρ_2 occupies a semi-infinite plane of $y < 0$.

Assume that the initial moment of disturbance is in the form of $\cos kx$. To guarantee that Φ_i satisfies the Laplace equation and the corresponding infinity disappearance condition, we choose

$$\eta = a\mathrm{e}^{nt}\cos kx, \quad \Phi_1 = A\mathrm{e}^{nt-ky}\cos kx, \quad \Phi_2 = B\mathrm{e}^{nt+ky}\cos kx. \tag{2.61}$$

Substituting these into (2.60), we get

$$\begin{aligned} &\rho_1(-kA - na) = \rho_2(kB - na), \\ &\rho_1(nA + V_1kA + ga) = \rho_2(nB - V_2kB + ga), \\ &-kA - V_1ka = -kB - V_2ka. \end{aligned}$$

This is a homogeneous linear algebraic equation for a, A and B. The condition for this non-trivial solution is that the corresponding coefficient determinant is zero, thus we can obtain the following dispersion relation:

$$n = \pm\sqrt{\frac{\rho_1+\rho_2}{\rho_1-\rho_2}kg - \frac{\rho V_1(V_2-V_1)k^2}{\rho_1-\rho_2}}. \tag{2.62}$$

Considering that (2.55), we have

$$n = \pm\sqrt{\frac{\rho_1+\rho_2}{\rho_1-\rho_2}kg - \frac{\Delta p^{(0)}k^2}{\rho_1-\rho_2}}. \tag{2.63}$$

Here $\Delta p^{(0)} = p_1^{(1)} - p_2^{(0)} = \rho_1 V_1(V_2 - V_1)$ is the ablation pressure. We have previously assumed the area "1" is an ablated medium, so $\rho_1 > \rho_2$. By (2.55), $|v_2| > |v_1|$ and V_i are negative, the ablation pressure is positive. When $g > 0$, that is the accelerating ablation, instability may occur, but when $g < 0$, n is pure, that is the deceleration ablation is stable.

Next we discuss a few special cases:

(1) When $\rho_1 \gg \rho_2$, (2.60) can be written as

$$n = \pm\sqrt{kg - \frac{\Delta p^{(0)}}{\rho_1}k^2}, \tag{2.64}$$

where $g > 0$ corresponds to the RT instability case, but slows the development of instability due to the presence of ablation pressure. When $\Delta p^{(0)} = \rho_1 g/k$, the instability can be truncated.

(2) When $\rho_1 \sim \rho_2$, (2.63) can be written as

$$n = \pm\sqrt{\frac{\rho_1 + \rho_2}{\rho_1 - \rho_2}kg - V_1^2 k^2}.$$

When $g > 0$, the instability develops rapidly. When $g < 0$, the section is quickly oscillated.

Now consider that the ablation product area is a swirling flow, and in the case of a spin in the "2" area, we choose

$$u_2^{(1)} = \Phi_{2x} + \Psi_{2x}, \quad v_2^{(1)} = \Phi_{2y} - \Psi_{2x}.$$

The "1" area is still non-rotating. There are four physical quantities to be determined, which are η, Φ_1, Φ_2, and Ψ_2. But the boundary condition (2.56) contains only three equations. The problem is uncertain. Landau assumed the normal velocity is zero when discussing the combustion

$$\eta_t = v_1^{(1)} = v_2^{(1)}.$$

There is a supplementary condition from (2.56)

$$p_1^{(1)} = p_2^{(1)}.$$

To get the expression (2.59) for the perturbation pressure $p_2^{(1)}$ in the "2" area, we must take

$$\Phi_{2t} + V_2\Psi_{2y} = 0.$$

Thus the momentum equation can be satisfied. If we take $\Psi_2 \sim \mathrm{e}^{kt+m_2 y}\sin kx$, then $m_2 = -n/V_2$.

Note that $V_2 < 0$, so $m_2 > 0$. Therefore Φ_2 satisfies the disappearing condition as $y \to -\infty$. Set

$\eta = a\mathrm{e}^{nt}\cos kx, \quad \Phi_1 = A\mathrm{e}^{nt-ky}\cos kx, \quad \Phi_2 = B\mathrm{e}^{nt+ky}\cos kx, \quad \Psi_2 = c\mathrm{e}^{nt+m_2 y}\sin kx.$

Substituting these into the boundary condition of $y = \eta(x,t)$ and expanding on y, we can get four constants which are determined relative to a, A, B and c. These four constants satisfy homogeneous linear algebraic equations, and the existence condition is that the corresponding coefficient determinant is zero. Thus,

$$n = \frac{\rho_\gamma k V_1}{\rho_1+\rho_2} \pm \sqrt{\frac{\rho_1-\rho_2}{\rho_1+\rho_2}kg + \frac{\rho_1 V_1 k^2}{\rho_1+\rho_2}\Big(V_2 - \frac{\rho_1 V_1}{\rho_1+\rho_2}\Big)}. \tag{2.65}$$

Since $V_1 < 0,\ V_2 < 0$, when $\rho_1 \gg \rho_2$, the second item in (2.65) square brackets can be omitted, we get

$$n = -\frac{\rho_1 k|V_1|}{\rho_1+\rho_2} \pm \sqrt{\frac{\rho_1-\rho_2}{\rho_1+\rho_2}kg + \frac{\rho_1 V_1 V_2 k^2}{\rho_1+\rho_2}}. \tag{2.66}$$

$g = 0$ is the combustion instability case discussed by Landau, If $\rho_1 \gg \rho_2$,

$$n = -k|V_1| \pm \sqrt{kg + V_1 V_2 k^2}.$$

When the burning speed is $v_\gamma = -V_1$, there is

$$n \sim \sqrt{kg + V_1 V_2 k^2}.$$

At this time, the instability is more serious. When k is a small constant, that is, for long waves, the term $V_1 V_2 k^2$ is ignored, there is

$$n = -k|V_1| + \sqrt{kg}.$$

Storm used a similar formula and multiplied a correction factor to estimate the instability of the laser ablation interface.

2.7 RT instability of magnetic field in MHD

As is shown in Figure 2.5, the ideal magnetohydrodynamics equations can be expressed as:

$$\frac{\partial \rho}{\partial t} + \rho(\nabla\cdot u) = 0, \tag{2.67}$$

$$\rho\frac{\mathrm{d}u}{\mathrm{d}t} + \nabla p = \rho g + \frac{1}{\mu_0}(\nabla\times B)\times B, \tag{2.68}$$

$$\nabla\times E = -\frac{\partial B}{\partial t}, \tag{2.69}$$

$$E = -u\times B. \tag{2.70}$$

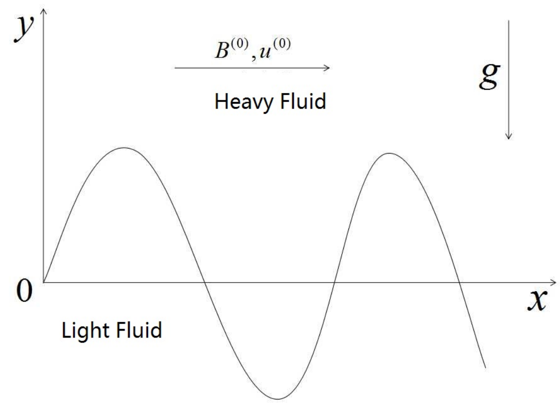

Figure 2.5

Linearizing (2.67)-(2.70) and expressing as the “0” order and the “1” order small perturbation scale, there are

$$\rho = \rho^{(0)} + \rho^{(1)}, \quad u = u^{(0)} + u^{(1)}, \quad p = p^{(0)} + p^{(1)}, \quad B = B^{(0)} + B^{(1)}.$$

Substituting these into (2.67)-(2.70), we obtain the linearized equations

$$\frac{\partial \rho^{(1)}}{\partial t} + u^{(0)} \cdot \nabla \rho^{(1)} + u^{(1)} \cdot \nabla \rho^{(0)} = 0, \tag{2.71}$$

$$\rho^0 \Big(\frac{\partial u^{(1)}}{\partial t} + u^0 \cdot \nabla u^{(1)} + u^{(1)} \cdot \nabla u^{(0)} \Big) + \rho^{(1)} \Big(\frac{\partial u^{(0)}}{\partial t} + u^{(0)} \cdot \nabla u^{(0)} \Big)$$
$$= \rho^{(1)} g - \nabla \Big(p^{(1)} + \frac{B^{(0)} \cdot B^{(1)}}{\mu_0} \Big) + \frac{1}{\mu_0} (B^{(0)} \cdot \nabla B^{(1)} + B^{(1)} \cdot \nabla B^{(0)}), \tag{2.72}$$

$$\frac{\partial B^{(1)}}{\partial t} = B^{(0)} \cdot \nabla u^{(1)} + B^{(1)} \cdot \nabla u^{(0)} - u^{(0)} \cdot \nabla B^{(1)} - u^{(1)} \cdot \nabla B^{(0)}. \tag{2.73}$$

In additional, using the incompressible conditions

$$\nabla \cdot u^{(1)} = 0. \tag{2.74}$$

Use the following initial flow field distribution

$$u^{(0)} = 0, \quad B^0 = B^0(y) e_x, \quad g^{(0)} = -g e_y,$$

and set $u^{(1)} = u^{(1)} e_x + v^{(1)} e_y$, $B^{(1)} = B^{(1)} e_x + C^{(1)} e_y$. Substituting the above initial fluid distribution and small perturbation into equations (2.71)-(2.73), there is

$$0 = \frac{\partial \rho^{(1)}}{\partial t} + v^{(1)} \frac{\mathrm{d}\rho^{(0)}}{\mathrm{d}y} + u^{(0)} \frac{\partial \rho^{(1)}}{\partial x}, \tag{2.75}$$

$$\rho^{(0)} \Big(\frac{\partial u^{(1)}}{\partial t} + u^{(0)} \frac{\partial u^{(1)}}{\partial x} + v^{(1)} \frac{\mathrm{d}u^{(0)}}{\mathrm{d}y} \Big) = -\frac{\partial p^{(1)}}{\partial x} + \frac{C^{(1)}}{\mu_0} \frac{\mathrm{d}B^{(0)}}{\mathrm{d}y}, \tag{2.76}$$

$$\rho^{(0)}\Big(\frac{\partial v^{(1)}}{\partial t}+u^{(0)}\frac{\partial v^{(1)}}{\partial x}\Big)=-\rho^{(1)}g-\frac{\partial p^{(1)}}{\partial y}-\frac{B^{(0)}}{\mu_0}\frac{\partial B^{(1)}}{\partial y}-\frac{B^{(1)}}{\mu_0}\frac{\mathrm{d}B^{(0)}}{\mathrm{d}y}+\frac{B^{(0)}}{\mu_0}\frac{\partial C^{(1)}}{\partial x}, \tag{2.77}$$

$$\frac{\partial B^{(1)}}{\partial t}=B_0\frac{\partial u^{(1)}}{\partial x}+C^{(1)}\frac{\mathrm{d}u^{(0)}}{\mathrm{d}y}-v^{(1)}\frac{\mathrm{d}B^{(0)}}{\mathrm{d}y}-u^{(0)}\frac{\partial B^{(1)}}{\partial x}, \tag{2.78}$$

$$\frac{\partial C^{(1)}}{\partial t}=B^{(0)}\frac{\partial v^{(1)}}{\partial x}-u^{(0)}\frac{\partial C^{(1)}}{\partial x}, \tag{2.79}$$

$$0=\frac{\partial u^{(1)}}{\partial x}+\frac{\partial v^{(1)}}{\partial y}. \tag{2.80}$$

Setting

$$\big(\rho^{(1)},u^{(1)},v^{(1)},p^{(1)},B^{(1)},C^{(1)}\big)=\big(\overline{\rho}(y),\overline{u}(y),\overline{v}(y),\overline{p}(y),\overline{B}(y),\overline{C}(y)\big)\mathrm{e}^{ikx+nt}. \tag{2.81}$$

Substituting (2.81) into (2.75)-(2.80) gives $u^{(0)}=0$ and

$$0=n\overline{\rho}+\overline{v}\frac{\mathrm{d}\rho^{(0)}}{\mathrm{d}y}, \tag{2.82}$$

$$n\rho^{(0)}\overline{u}=-ik\overline{p}+\frac{\overline{C}}{\mu_0}\frac{\mathrm{d}B^{(0)}}{\mathrm{d}y}, \tag{2.83}$$

$$n\rho^{(0)}\overline{v}=-\overline{\rho}g-\frac{\mathrm{d}\overline{p}}{\mathrm{d}y}-\frac{B^{(0)}}{\mu_0}\frac{\mathrm{d}\overline{B}}{\mathrm{d}y}-\frac{\overline{B}}{\mu_0}\frac{\mathrm{d}B^{(0)}}{\mathrm{d}y}+ik\frac{B^{(0)}}{\mu_0}\overline{C}, \tag{2.84}$$

$$n\overline{B}=ikB^{(0)}\overline{u}-\overline{v}\frac{\mathrm{d}B^{(0)}}{\mathrm{d}y}, \tag{2.85}$$

$$n\overline{C}=ikB^{(0)}\overline{v}, \tag{2.86}$$

$$0=ik\overline{u}+\frac{\mathrm{d}\overline{v}}{\mathrm{d}y}, \tag{2.87}$$

where $n=\gamma-iw$, γ represents the linear growth rate, and $w=w(k)$ is the disturbance frequency.

The second-order eigenvalue equation for the perturbation velocity $\overline{v}$ can be obtained by simplifying the above algebraic equations

$$\begin{aligned}&n^2\rho^{(0)}\frac{\mathrm{d}^2\overline{v}}{\mathrm{d}y^2}+n^2\frac{\mathrm{d}\rho^{(0)}}{\mathrm{d}y}\frac{\mathrm{d}\overline{v}}{\mathrm{d}y}-k^2n^2\rho^{(0)}\overline{v}+k^2g\frac{\mathrm{d}\rho^{(0)}}{\mathrm{d}y}\overline{v}\\&=\frac{1}{\mu_0}\Big[-k^2(B^{(0)})^2\frac{\mathrm{d}^2\overline{v}}{\mathrm{d}y^2}-2k^2B^{(0)}\frac{\mathrm{d}B^{(0)}}{\mathrm{d}y}\frac{\mathrm{d}\overline{v}}{\mathrm{d}y}+k^4(B^{(0)})^2\overline{v}\Big].\end{aligned} \tag{2.88}$$

Integrating (2.88) from $-\infty$ to ∞, we can obtain the following simple form:

$$-n^2k\int_{-\infty}^{\infty}\rho^{(0)}\overline{v}\mathrm{d}y+k^2g\int_{-\infty}^{\infty}\frac{\mathrm{d}\rho^{(0)}}{\mathrm{d}y}\overline{v}\mathrm{d}y=\frac{k^4}{\mu_0}\int_{-\infty}^{\infty}(B^{(0)})^2\overline{v}\mathrm{d}y. \tag{2.89}$$

Set

$$\overline{v} = \overline{v}_{\max}\mathrm{e}^{-k|y|} = \begin{cases} \overline{v}_{\max}\mathrm{e}^{-ky}, & y > 0, \\ \overline{v}_{\max}\mathrm{e}^{ky}, & y < 0, \end{cases} \qquad k = \frac{2\pi}{\lambda}, \tag{2.90}$$

$$\rho^{(0)} = \begin{cases} \rho_1 - \dfrac{\rho_1 - \rho_2}{2}\mathrm{e}^{-\frac{y}{2L_p}}, & y > 0, \\ \rho_2 + \dfrac{\rho_1 - \rho_2}{2}\mathrm{e}^{\frac{y}{2L_p}}, & y < 0, \end{cases} \tag{2.91}$$

$$B^{(0)} = \begin{cases} B_1 - \dfrac{B_1 - B_2}{2}\mathrm{e}^{-\frac{y}{2L_B}}, & y > 0, \\ B_2 + \dfrac{B_1 - B_2}{2}\mathrm{e}^{\frac{y}{2L_B}}, & y < 0, \end{cases} \tag{2.92}$$

where L_p and L_B are the density and the magnetic field filter layer widths, respectively.

$$n^2 = \frac{\Lambda_T kg}{kL_\rho + 1} - \frac{k^2}{\mu_0(\rho_1 + \rho_2)}\Big[B_1^2 + B_2^2 - \frac{(B_1 - B_2)^2}{1 + 1/kL_B} + \frac{(B_1 - B_2)^2}{2 + 4/kK_B}\Big], \tag{2.93}$$

where $\Lambda_T = (\rho_1 - \rho_2)/(\rho_1 + \rho_2)$ is the Atwood number.

3 Some Mathematical Analysis Theory Results

In this section we mainly introduce some theoretical analysis results of Guo Yan and his collaborators.

3.1 Linear instability of compressible viscous fluids

As is shown in Figure 3.1, we consider a horizontal infinite flat area $\Omega = \mathbf{R}^2 \times (-m, l) \subset \mathbf{R}^3$.

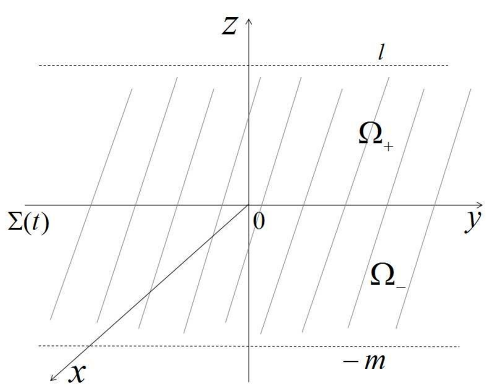

Figure 3.1

Consider the compressible NS equation:

$$\begin{cases} \partial_t \rho_\pm + \mathrm{div}(\rho_\pm u_\pm) = 0 \\ \rho_\pm(\partial_t u_\pm + u_\pm \cdot \nabla u_\pm) + \mathrm{div} S_\pm = -g\rho_\pm \vec{e}_3, \end{cases} \tag{3.1}$$

where

$$S_\pm = P_\pm(\rho_\pm I) - \varepsilon_\pm(\rho_\pm)\Big(Du_\pm + Du_\pm^{\mathrm{T}} - \frac{2}{3}\mathrm{div}u_\pm I\Big) - \delta_\pm(\rho_\pm)\mathrm{div}u_\pm I,$$

under the jump condition on the interface

$$\begin{cases}(u_+)|_{\Sigma(t)} - (u_-)|_{\Sigma(t)} = 0,\\ (S_+\cdot\nu)|_{\Sigma(t)} - (S_-\cdot\nu)|_{\Sigma(t)} = \sigma H\cdot\nu,\end{cases} \tag{3.2}$$

where $\sigma \geq 0$, H is the mean curvature of the surface.

The boundary is

$$u_-(x_1, x_2, -m, t) = u_+(x_1, x_2, l, t) = 0, \quad \text{for any } (x_1, x_2) \in \mathbf{R}^2,\ t > 0. \tag{3.3}$$

Set the density and the speed to be periodic on the space $\Omega_\pm(t)$,

$$\begin{cases}\rho_\pm(x + 2\pi Lk_1e_1 + 2\pi Lk_2e_2, t) = \rho_\pm(x,t), \quad x \in \Omega_\pm(t),\\ u_\pm(x + 2\pi Lk_1e_1 + 2\pi Lk_2e_2, t) = u_\pm(x,t), \quad x \in \Omega_\pm(t).\end{cases} \tag{3.4}$$

In the Lagrange coordinates, we can rewritten the equations as another form. Thus we can solve this equations in the fixed domain.

Setting $\Omega_- = \mathbf{R}^2 \times (-m, 0)$, $\Omega_+ = \mathbf{R}^2 \times (0, l)$, there exists a mapping:

$$\eta_\pm : \Omega_\pm \to \Omega_\pm(0), \tag{3.5}$$

such that $\Sigma_0 = \eta_\pm^0\{x_3 = 0\}$, $\eta_+^0 = \{x_3 = l\}$, $\eta_-^0 = \{x_3 = -m\}$.

Define a flow mapping $\eta_\pm$:

$$\begin{cases}\partial_t\eta_\pm(x,t) = u_\pm(\eta_\pm(x,t),t),\\ \eta(x,0) = \eta_\pm^0(x),\end{cases} \tag{3.6}$$

where $\Omega_\pm(t) = \eta_\pm(\Omega_\pm, t)$, $\Sigma(t) = \eta_\pm\{\{x_3 = 0\}, t\}$.

Defining the Lagrange unknowns

$$\begin{cases}v_\pm(x,t) = u_\pm(\eta_\pm(x,t),t),\\ q_\pm(x,t) = \rho_\pm(\eta_\pm(x,t),t).\end{cases} \tag{3.7}$$

By introducing $\eta_\pm, v_\pm, q_\pm$, $A^{\mathrm{T}} = (D\eta)^{-1}$. In Lagrange coordinates, the evolution equations for v, q, η are

$$\begin{cases}\partial_t\eta_i = v_i,\\ \partial_t q + qA_{ij}\partial_j v_i = 0,\\ q\partial_t v_i + A_{jk}\partial_k T_{ij} = -gqA_{ij}\partial_j\eta_3,\end{cases} \tag{3.8}$$

where the viscous stress tensor T is

$$T_{ij} = p(q)I_{ij} - \varepsilon(q)\Big(A_{jk}\partial_k v_i + A_{ik}\partial_k v_j - \frac{2}{3}(A_{lk}(\partial_k v_l)I_{ij})\Big) - \delta(q)(A_{lk}\partial_k v_l)I_{ij} \tag{3.9}$$

with I_{ij} being an element of 3×3 matrix I.

To obtain the jump conditions, for a quantity $f=f_{\pm}$, we define the interfacial jump as

$$[[f]]:=f_+|_{x_3=0}-f_-|_{x_3=0}. \tag{3.10}$$

The jump conditions across the interface are

$$[[v]]=0, \tag{3.11}$$

$$[[Tn]]=\sigma Hn, \tag{3.12}$$

where

$$n=\frac{\partial_1\eta\times\partial_2\eta}{|\partial_1\eta\times\partial_2\eta|_{x_3=0}}, \tag{3.13}$$

for the unit normal to the surface $\Sigma(t)=\eta\,(x_3=0,t)$ and H for twice the mean curvature of $\Sigma(t)$,

$$H=\Big(\frac{|\partial_1\eta|^2\partial_2^2\eta-2(\partial_1\eta\cdot\partial_2\eta)\partial_1\partial_2\eta+|\partial_2\eta|^2\partial_1^2\eta}{|\partial_1\eta|^2|\partial_2\eta|^2-|\partial_1\eta\cdot\partial_2\eta|^2}\Big)\cdot n. \tag{3.14}$$

Finally, we require the no-slip boundary condition

$$v_-(x_1,x_2,-m,t)=v_+(x_1,x_2,l,t)=0. \tag{3.15}$$

In the following we seek a steady-state solution, by taking

$$\begin{aligned}&v=0,\quad \eta=Id,\quad q(x,t)=\rho_0(x_3),\\&\eta(\{x_3=0\})=\{x_3=0\},\quad H=0,\quad n=e_3,\quad A=I.\end{aligned}$$

We deduce the following ODE

$$\frac{\mathrm{d}p(\rho_0)}{\mathrm{d}x_3}=-g\rho_0, \tag{3.16}$$

subject to the jump condition

$$[[p(\rho_0)]]=0. \tag{3.17}$$

To solve this we introduce the enthalpy function defined by

$$h_{\pm}(z)=\int_1^z\frac{P'_{\pm}(r)}{r}\mathrm{d}r. \tag{3.18}$$

To discuss the Rayleigh-Taylor instability, we must prove the fluid is denser above the interface, that is $\rho_0^+>\rho_0^-$, then

$$[[\rho_0]]=\rho_0^+-\rho_0^->0. \tag{3.19}$$

The solution is then given by $P_{\pm}(\rho)=K_{\pm}\rho^{\gamma}$, $K_{\pm}>0$, $\gamma_{\pm}\geq 1$,

$$\rho_0(x_3)=\begin{cases}\Big((\rho_0^-)^{\gamma_--1}-\dfrac{g(\gamma_--1)}{K_-\gamma_-}x_3\Big)^{\frac{1}{\gamma_--1}}, & x_3<0,\\ \Big((\rho_0^+)^{\gamma_+-1}-\dfrac{g(\gamma_+-1)}{K_+\gamma_+}x_3\Big)^{\frac{1}{\gamma_+-1}}, & 0<x_3<\dfrac{K_+\gamma_+}{g(\gamma_+-1)}(\rho_0^+)^{\gamma_+-1},\\ 0, & x_3>\dfrac{K_+\gamma_+}{g(\gamma_+-1)}(\rho_0^+)^{\gamma_+-1}.\end{cases} \tag{3.20}$$

For a polytropic gas law, $\rho_0^+>\rho_0^-$ is equivalent to the conditions

$$\Big(\frac{K_-}{K_+}\Big)^{\frac{1}{\gamma_+}}(\rho_0^-)^{\frac{\gamma_-}{\gamma_+}}>\rho_0^-,\quad \text{if and only if}\quad (\rho_0^-)^{\gamma_--\gamma_+}>\frac{K_+}{K_-}. \tag{3.21}$$

If $\gamma_+=\gamma_-$, for any $\rho_0^->0$, $K_->K_+$ is required. If $\gamma_+\neq\gamma_-$, then $K_-,K_+>0$ can be arbitrary, but we must require that $\rho_0^->0$ must satisfy

$$\begin{cases}\rho_0^->\Big(\dfrac{K_+}{K_-}\Big)^{\frac{1}{(\gamma_--\gamma_+)}}, & \gamma_->\gamma_+,\\ \rho_0^-<\Big(\dfrac{K_-}{K_+}\Big)^{\frac{1}{(\gamma_+-\gamma_-)}}, & \gamma_+>\gamma_-.\end{cases}$$

We now linearize the nonlinear problem (3.8) around the steady-state solution $v=0$, $\eta=Id$, $q=\rho_0$. The obtained linearized equations with η, v, q are

$$\begin{cases}\partial_t\eta=v,\\ \partial_t q+\rho_0\mathrm{div}v=0\end{cases} \tag{3.22}$$

and

$$\rho_0\partial_t v+\nabla(P'(\rho_0)q)+g\rho e_3+g\rho_0\nabla\eta_3=\mathrm{div}\Big(\varepsilon_0(Dv+Dv^{\mathrm{T}}-\frac{2}{3}(\mathrm{div}v)I)+\delta_0(\mathrm{div}v)I\Big), \tag{3.23}$$

where $\varepsilon_0=\varepsilon_0(\rho_0)$, $\delta_0=\delta_0(\rho_0)$.

The jump conditions is linearized to

$$\begin{cases}[[v]]=0,\\ \Big[\Big[P'(\rho_0)qI-\varepsilon_0(Dv+Dv^{\mathrm{T}})-\Big(\delta_0-\dfrac{2\varepsilon_0}{3}\Big)(\mathrm{div}v)I\Big]\Big]e_3=\sigma\Delta_{x_1,x_2}\eta_3e_3,\end{cases} \tag{3.24}$$

under the initial data $\eta(0)=\eta_0,\ v(0)=v_0,\ q(0)=q_0$, that satisfy the jump and boundary conditions in addition to the assumption that $[[\eta_0]]=0$, which implies that $\eta(t)$ is continuous across $x_3=0$, for any $t\geq 0$.

3.2 Growing mode ansatz

We will look for a growing normal mode solution to (3.22), (3.23) by first assuming an ansatz

$$v(x,t)=w(x)\mathrm{e}^{\lambda t},\quad q(x,t)=\overline{q}(x)\mathrm{e}^{\lambda t},\quad \eta(x,t)=\overline{\eta}(x)\mathrm{e}^{\lambda t}, \tag{3.25}$$

for some $\lambda > 0$. Plugging the ansatz into (3.22)-(3.23), we may solve the first and second equations for $\overline{q}, \overline{\eta}$ in terms of v. By doing so and eliminating them from the third equation, we arrive at the time-invariant equation

$$\begin{aligned}&\lambda^2\rho_0 w - \nabla(P'(\rho_0)\rho_0 \mathrm{div} w) + g\rho_0\nabla w_3 - g\rho_0 \mathrm{div} w e_3\\ =&\mathrm{div}\Big(\lambda\varepsilon_0\Big(Dw + Dw^{\mathrm{T}} - \frac{2}{3}(\mathrm{div} w)I\Big) + \lambda\delta_0(\mathrm{div} w)I\Big).\end{aligned} \tag{3.26}$$

This is coupled to the jump conditions $[[w]] = 0$ and

$$\Big[\Big[\lambda\delta_0 - \frac{2}{3}\lambda\varepsilon_0 + P'(\rho_0)\mathrm{div} wI + \lambda\varepsilon_0(Dw + Dw^{\mathrm{T}})\Big]\Big] = -\sigma\Delta_{x_1,x_2} w_3 e_3, \tag{3.27}$$

with the boundary conditions $w_-(x_1, x_2, -m) = w_+(x_1, x_2, l) = 0$. Notice that the first jump condition implies the assumptions on

$$\eta(0) = \overline{\eta}(0) = \frac{w(0)}{\lambda}.$$

Since the coefficients of the linear problem (3.26) depend only on the x_3 variable, we define

$$w_1(x) = -i\varphi(x_3)\mathrm{e}^{ix'\cdot\xi}, \quad w_2(x) = -i\theta(x_3)\mathrm{e}^{ix'\cdot\xi}, \quad w_3 = \psi(x_3)\mathrm{e}^{ix'\cdot\xi}, \tag{3.28}$$

$$\mathrm{div} w = (\xi_1\varphi + \xi_2\theta + w_3')\mathrm{e}^{ix'\cdot\xi}, \tag{3.29}$$

$$Dw + Dw^{\mathrm{T}} = \begin{pmatrix} 2\xi_1\varphi & \xi_1\theta + \xi_2\varphi & i(\xi_1\psi - \varphi') \\ \xi_1\theta + \xi_2\varphi & 2\xi_2\theta & i(\xi_2\psi - \varphi') \\ i(\xi_1\psi - \varphi') & i(\xi_2\psi - \theta') & -2\psi' \end{pmatrix} \mathrm{e}^{ix'\cdot\xi}. \tag{3.30}$$

For each fixed ξ, we arrive at the following system of ODEs for $\varphi(x_3), \theta(x_3), \psi(x_3)$ and λ,

$$\begin{aligned}&-(\lambda\varepsilon_0\varphi')' + \Big[\lambda^2\rho_0 + \lambda\varepsilon_0|\xi|^2 + \xi_1^2(\lambda_0\delta_0 + \frac{1}{3}\lambda\varepsilon_0 + P'(\rho_0)\rho_0)\Big]\varphi\\ =&-\xi_1\Big[\Big(\lambda\delta_0+\frac{1}{3}\lambda\varepsilon_0+P'(\rho_0)\rho_0\Big)\psi'+(\lambda\varepsilon_0'-g\rho_0)\psi\Big]-\xi_1\xi_2\Big[\lambda\delta_0+\frac{1}{3}\lambda\varepsilon_0+P'(\rho_0)\rho_0\Big]\theta,\end{aligned} \tag{3.31}$$

$$\begin{aligned}&-(\lambda\varepsilon_0\theta')' + \Big[\lambda^2\rho_0 + \lambda\varepsilon_0|\xi|^2 + \xi_2^2(\lambda_0\delta_0 + \frac{1}{3}\lambda\varepsilon_0 + P'(\rho_0)\rho_0)\Big]\theta\\ =&-\xi_2\Big[\Big(\lambda\delta_0+\frac{1}{3}\lambda\varepsilon_0+P'(\rho_0)\rho_0\Big)\psi'+(\lambda\varepsilon_0'-g\rho_0)\psi\Big]-\xi_1\xi_2\Big[\lambda\delta_0+\frac{1}{3}\lambda\varepsilon_0+P'(\rho_0)\rho_0\Big]\varphi,\end{aligned} \tag{3.32}$$

$$\begin{aligned}&-\Big[\Big(\frac{4\lambda\varepsilon_0}{3} + \lambda\delta_0 + P'(\rho_0)\rho_0\Big)\psi'\Big]' + (\lambda^2\rho_0 + \lambda\varepsilon_0|\xi|^2)\psi\\ =&\Big[\lambda\delta_0 + \frac{1}{3}\lambda\varepsilon_0 + P'(\rho_0)(\xi_1\varphi + \xi_2\theta)\Big]' + (g\rho_0 - \lambda\varepsilon_0')(\xi_1\varphi + \xi_2\theta).\end{aligned} \tag{3.33}$$

The first jump condition yields

$$[[\varphi]] = [[\theta]] = [[\psi]] = 0. \tag{3.34}$$

The second jump condition becomes

$$\left[\left[\Big(\lambda\delta_0 - \frac{2}{3}\lambda\varepsilon_0 + P'(\rho_0)\Big)(\psi' + \xi_1\varphi + \xi_2\theta)e_3 + \lambda\varepsilon_0 \begin{pmatrix} i(\xi_1\psi - \varphi') \\ i(\xi_2\psi - \theta') \\ 2\psi' \end{pmatrix}\right]\right] = \sigma|\xi|^2\psi e_3, \tag{3.35}$$

which implies that

$$[[\lambda\varepsilon_0(\varphi' - \xi_1\psi)]] = [[\lambda\varepsilon_0(\theta' - \xi_2\psi)]] = 0, \tag{3.36}$$

and that

$$\left[\left[\Big(\lambda\delta_0 + \frac{1}{3}\lambda\varepsilon_0 + P'(\rho_0)\rho_0\Big)(\psi' + \xi_1\varphi + \xi_2\theta)\right]\right] + [[\lambda\varepsilon_0(\psi' - \xi_1\varphi - \xi_2\theta)]] = \sigma|\xi|^2\psi. \tag{3.37}$$

The boundary conditions

$$\varphi(-m) = \varphi(l) = \theta(-m) = \theta(l) = \psi(-m) = \psi(l) = 0 \tag{3.38}$$

must also hold. If (φ, θ, ψ) is a solution to (3.31)-(3.33) for $\xi \in R^2$ and λ, then for any rotation operator $R \in SO(2)$, $(\widetilde{\psi}, \widetilde{\theta}) = R(\psi, \theta)$ is a solution to the same equations for $\widetilde{\xi} = R\xi$ with ψ, λ unchanged. So, by choosing an appropriate rotation, we may assume without loss of generality that $\xi_2 = 0$, and $\xi_1 = |\xi|$. In this setting θ solves

$$\begin{cases} -(\lambda\varepsilon_0\theta')' + (\lambda^2\rho_0 + \varepsilon_0|\xi|^2)\theta = 0, \\ \theta(-m) = \theta(l) = 0, \\ [[\theta]] = [[\lambda\varepsilon\theta']] = 0. \end{cases} \tag{3.39}$$

Multiplying this equation by θ, integrating over $(-m, l)$, integrating by parts, and using the jump conditions yield

$$\int_{-l}^{m} \lambda\varepsilon_0|\theta'|^2 + (\lambda^2\rho_0 + \lambda\varepsilon_0|\xi|^2)\theta^2 = 0, \tag{3.40}$$

which implies that $\theta = 0$, since $\lambda > 0$. This reduces to the pair of equations for φ, ψ:

$$\begin{aligned} -\lambda^2\rho_0\varphi = & -(\lambda\varepsilon_0\varphi')' + |\xi|^2\Big(\frac{4}{3}\lambda\varepsilon_0 + \frac{1}{3}\lambda\delta_0 + P'(\rho_0)\rho_0\Big)\varphi \\ & + |\xi|\Big[\Big(\lambda\delta_0 + \frac{1}{3}\lambda\varepsilon_0 + P'(\rho_0)\rho_0\Big)\psi' + (\lambda\varepsilon_0' - g\rho_0)\psi\Big], \end{aligned} \tag{3.41}$$

$$\begin{aligned} -\lambda^2\rho_0\psi = & -\Big[\Big(\frac{4\lambda\varepsilon_0}{3} + \lambda\delta_0 + P'(\rho_0)\rho_0\Big)\psi'\Big]' + \lambda\varepsilon_0|\xi|^2\psi \\ & - |\xi|\Big[\Big(\lambda\delta_0 + \frac{1}{3}\lambda\varepsilon_0 + P'(\rho_0)\Big)\varphi' + (g\rho_0 - \lambda\varepsilon_0')\varphi\Big], \end{aligned} \tag{3.42}$$

along with the jump conditions

$$[[\varphi]] = [[\psi]] = [[\lambda\varepsilon_0(\varphi' - |\xi|\psi)]] = 0, \tag{3.43}$$

$$\left[\left[\Big(\lambda\delta_0 + \frac{1}{3}\lambda\varepsilon_0 + P'(\rho_0)\rho_0\Big)(\psi' + |\xi|\varphi)\right]\right] + [[\lambda\varepsilon_0(\varphi') - |\xi|\psi]] = \sigma|\xi|^2\psi, \tag{3.44}$$

and the boundary conditions

$$\varphi(-m) = \varphi(l) = \psi(-m) = \psi(l) = 0. \tag{3.45}$$

3.3 Statement of main results

In the absence of viscosity, $\varepsilon = \delta = 0$ and for a fixed spatial frequency $\xi \neq 0$, (3.41), (3.42) can be viewed as an eigenvalue problem with eigenvalue $-\lambda^2$. Such a problem has a natural variational structure that allows for the construction of solutions via the direct methods and for a variational characterization of the eigenvalue according to

$$-\lambda^2 = \inf \frac{E(\varphi,\psi)}{J(\varphi,\psi)}, \tag{3.46}$$

where

$$E(\varphi,\psi) = \frac{1}{2}\int_{-m}^{l} P'(\rho_0)\rho_0(\psi' + |\xi|\varphi)^2 - 2g\rho_0|\xi|\psi\varphi, \tag{3.47}$$

and

$$J(\varphi,\psi) = \frac{1}{2}\int_{-m}^{l} \rho_0(\varphi^2 + \psi^2). \tag{3.48}$$

Unfortunately, when viscosity is present, the natural variational structure breaks down since λ^2 appears quadratically as a multiplier of ρ_0 and linearly as a multiplier of ε_0 and δ_0. Since the equations imply a quadratic relationship between λ and various integrals of the solution, they can be solved for λ to determine the sign of Reλ. On the other hand, the appearance of λ both quadratically and linearly eliminates the capacity by using constrained minimization techniques to produce solutions to the equations.

In order to circumvent this problem and restore the ability to use variational methods, we artificially remove the linear dependence on λ. To this end, we define the modified viscosities $\widetilde{\varepsilon} = s\varepsilon_0$, $\widetilde{\delta} = s\delta$, where $s > 0$ is an arbitrary parameter. We then introduce a family of modified problems given by

$$\begin{aligned}
-\lambda^2\rho_0\varphi = {} & -(\widetilde{\varepsilon}\varphi')' + |\xi|^2\Big(\frac{4}{3}\lambda\widetilde{\varepsilon} + \widetilde{\delta}P'(\rho_0)\rho_0\Big)\varphi \\
& +|\xi|\Big[\Big(\widetilde{\delta} + \frac{1}{3}\widetilde{\varepsilon} + P'(\rho_0)\rho_0\Big)\psi' + (\widetilde{\varepsilon}' - g\rho_0)\psi\Big],
\end{aligned} \tag{3.49}$$

$$\begin{aligned}
-\lambda^2\rho_0\psi = {} & -\Big[\Big(\frac{4\widetilde{\varepsilon}}{3} + \widetilde{\delta} + P'(\rho_0)\rho_0\Big)\psi'\Big]' + \widetilde{\varepsilon}|\xi|^2\psi \\
& -|\xi|\Big[\Big(\Big(\widetilde{\delta} + \frac{\widetilde{\varepsilon}}{3} + P'(\rho_0)\rho_0\Big)\varphi\Big)' + (g\rho_0 - \widetilde{\varepsilon}')\varphi\Big],
\end{aligned} \tag{3.50}$$

along with the jump conditions

$$[[\varphi]] = [[\psi]] = [[\lambda\varepsilon_0(\varphi' - |\xi|\psi)]] = 0, \tag{3.51}$$

$$\Big[\Big[\Big(\widetilde{\delta} + \frac{1}{3}\widetilde{\varepsilon} + P'(\rho_0)\rho_0\Big)(\psi' + |\xi|\varphi)\Big]\Big] + [[\widetilde{\varepsilon}(\varphi') - |\xi|\psi]] = \sigma|\xi|^2\psi, \tag{3.52}$$

and the boundary conditions

$$\varphi(-m) = \varphi(l) = \psi(-m) = \psi(l) = 0. \tag{3.53}$$

A solution to the modified problem with $\lambda = s$ corresponds to a solution to the original problem.

Modifying the problem in this way restores the variational structure and allows us to apply a constrained minimization to the viscous analogue of the energy E defined above to find a solution to (3.49),(3.50) with $\lambda = \lambda(|s|, s) > 0$, when $s > 0$ is sufficiently small and precisely when

$$0 < |\xi| \leq |\xi|_c = \sqrt{\frac{g[[\rho_0]]}{\sigma}}. \tag{3.54}$$

We then further exploit the variational structure to show that λ is a continuous function and is strictly increasing in s. Using this, we show in Theorem 3.6 that the parameter s can be uniquely chosen so that

$$s = \lambda(|s|, s), \tag{3.55}$$

which implies that there exists a solution to the original problem (3.41),(3.42). This choice of s allows us to consider $\lambda = \lambda(|\xi|)$, which gives rise to a solution to system (3.32),(3.33) as well.

Theorem 3.1 *For $\xi \in \mathbb{R}^2$, so that $0 < |\xi|^2 < g[[\rho_0]]/\sigma$, there exists a solution $\varphi = \varphi(\xi, x_3)$, $\theta = \theta(\xi, x_3)$, $\psi = \psi(\xi, x_3)$ $\lambda = \lambda(\xi) > 0$ to (3.32),(3.33) satisfying the appropriate jump and boundary conditions so that $\psi = \psi(\xi, 0) \neq 0$. The solutions are smooth when restricted to $(-m, 0)$ or $(0, l)$, and they are equivariant in ξ in the sense that if $R \in SO(2)$, that is a rotation operator, then*

$$\begin{pmatrix} \varphi = \rho(R\xi, x_3) \\ \theta = \theta(R\xi, x_3) \\ \psi = \psi(R\xi, x_3) \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & 0 \\ R_{21} & R_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \varphi(\xi, x_3) \\ \theta(\xi, x_3) \\ \psi(\xi, x_3) \end{pmatrix}. \tag{3.56}$$

Without surface tension ($\sigma = 0$), it is possible to construct a solution to (3.55) with $\lambda > 0$, for any $\xi \neq 0$, but with surface tension ($\sigma > 0$) there is a critical frequency $|\xi|_c = \sqrt{g[[\rho_0]]/\sigma}$ for which there is no solution with $\lambda > 0$ being available if $|\xi| \geq |\xi|_c$. In the nonperiodic case, we capture a continuum $|\xi| \in (0, |\xi|_c)$ of growing mode solutions, but in the $2\pi L$ periodic case we find only finitely many solutions. Indeed, if

$$\sqrt{\frac{\sigma}{g[[\rho_0]]}} < L, \tag{3.57}$$

then there is a positive but finite number of spatial frequencies $\xi \in (L^{-1}Z)^2$ with $|\xi| < |\xi|_c$. On the other hand, if

$$L \leq \sqrt{\frac{\sigma}{g[[\rho_0]]}}, \tag{3.58}$$

then our method fails to construct any growing mode solutions at all. It is important to know the behavior of $\lambda(|\xi|)$ as $|\xi|$ varies within $0 < |\xi| < |\xi|_c$. It is easy to show that $\lambda(|\xi|)$ is continuous and satisfies

$$\lim_{|\xi|\to 0} \lambda(|\xi|) = \lim_{|\xi|\to|\xi|_c} \lambda(|\xi|) = 0. \tag{3.59}$$

In the nonperiodic case, this implies that there is a largest growth rate

$$0 < \Lambda = \max_{0\le|\xi|\le|\xi|_c} \lambda(|\xi|). \tag{3.60}$$

In the periodic case for L satisfying (3.57), the largest rate is always achieved and is given by

$$0 < \Lambda_L = \sup \big\{ \lambda(|\xi|) | \xi \in (L^{-1}Z)^2, |\xi| \in (0, |\xi|_c) \big\}. \tag{3.61}$$

Note that in general $\Lambda_L < \Lambda$.

The stabilizing effects of viscosity and surface tension are evident in these results. Without viscosity or surface tension, there is $\lambda(\xi) \to \infty, |\xi| \to \infty$. With viscosity but no surface tension, all spatial frequencies remain unstable, but the growth rate $\lambda(\xi)$ is bounded. With viscosity and surface tension, only a critical interval of spatial frequencies is unstable, and $\lambda(|\xi|)$ remains bounded. Finally, with viscosity, surface tension and the periodicity L satisfying (3.58), there do not exist any growing modes.

In the periodic case when L satisfies (3.57), the solutions to (3.32),(3.33) constructed in Theorem 3.1 immediately give rise to growing mode solutions to (3.22), (3.23).

Theorem 3.2 *Suppose that L satisfies* (3.57)*, and let $\xi_1, \xi_2 \in (L^{-1}E)^2$ be lattice points such that $\xi_1 = -\xi_2$, $\lambda(\xi_i) = \Lambda_L$, where Λ_L is defined by* (3.66)*. Define*

$$\widehat{w}(\xi, x_3) = -i\varphi(\xi, x_3)e_1 - i\theta(\xi, x_3)e_2 + \psi(\xi, x_3)e_3, \tag{3.62}$$

where φ, θ, ψ are the solutions provided by Theorem 3.1*. Writing $x' = x_1e_1 + x_2e_2$, we define*

$$\eta(x,t) = \mathrm{e}^{\Lambda_L t} \sum_{j=1}^{3} \widehat{w}(\xi_j, x_3)\mathrm{e}^{ix'\xi_j} \quad v(x,t) = \Lambda_L \mathrm{e}^{\Lambda_L t} \sum_{j=1}^{2} \widehat{w}(\xi_j, x_3)\mathrm{e}^{ix'\xi_j} \tag{3.63}$$

and

$$q(x,t) = \mathrm{e}^{-\Lambda_L t}\rho_0(x_3) \sum_{j=1}^{2} (e_i\xi_j\varphi(\xi_j, x_3)) + e_i\xi_j\theta(\xi_j, x_3) + \partial_3\psi(\xi_j, x_3))\mathrm{e}^{ix'\xi}. \tag{3.64}$$

Then η, v, q are real solutions to (3.22),(3.23) *with the corresponding jump and boundary conditions. For every $t > 0$, we have $\eta(t), v(t), q(t) \in H^k(\Omega)$, and*

$$\|\eta(t)\|_{H^k} = \mathrm{e}^{t\Lambda_L}\|\eta(0)\|_{H^k}, \quad \|v(t)\|_{H^k} = \mathrm{e}^{t\Lambda_L}\|v(0)\|_{H^k}, \quad \|q(t)\|_{H^k} = \mathrm{e}^{t\Lambda_L}\|q(0)\|_{H^k}. \tag{3.65}$$

Remark 3.1 In this theorem, the space $H^k(\Omega)$ is not the usual Sobolev space of order k, but what we call the piecewise Sobolev space of order k. In the nonperiodic case, although $\Lambda = \lambda(|\xi|)$ may be achieved for some $\xi \in R^2$, $|\xi| \in (0, |\xi|_c)$, no $L^2(\Omega)$ solution to (3.22),(3.23) may be constructed.

Theorem 3.3 *Let $f \in C_c^\infty(0, |\xi|_c)$ be a real-valued function. For $\xi \in R^2$ with $|\xi| \in (0, |\xi|_c)$ define*

$$\widehat{w}(\xi, x_3) = -i\varphi(\xi, x_3)e_1 - i\theta(\xi, x_3)e_2 + \psi(\xi, x_3)e_3, \tag{3.66}$$

where φ, θ, ψ are the solutions provided by Theorem 3.1. *Writing $x' = x_1e_1 + x_2e_2$ we define*

$$\eta(x,t) = \frac{1}{4\pi^2}\int_{R^2} f(|\xi|)\widehat{w}(\xi, x_3)\mathrm{e}^{\lambda(|\xi|)t}\mathrm{e}^{ix'\xi}\mathrm{d}\xi, \tag{3.67}$$

$$v(x,t) = \frac{1}{4\pi^2}\int_{R^2} \lambda(|\xi|)f(|\xi|)\widehat{w}(\xi, x_3)\mathrm{e}^{\lambda(|\xi|)t}\mathrm{e}^{ix'\xi}\mathrm{d}\xi, \tag{3.68}$$

$$q(x,t) = \frac{1}{4\pi^2}\int_{R^2} f(|\xi|)(\xi_1\varphi(\xi, x_3) + \xi_2\theta(\xi, x_3) + \partial_{x_3}\psi(\xi, x_3)\mathrm{e}^{\lambda(|\xi|)t}\mathrm{e}^{ix'\xi}\mathrm{d}\xi. \tag{3.69}$$

Then η, v, q are real-valued solutions to the linearized equations (3.22),(3.23) *along with the corresponding jump and boundary conditions. The solutions are equivariant in the sense that if $R \in SO(3)$ is a rotation that keeps the vector e_3 fixed, then*

$$\eta(Rx,t) = R\eta(x,t), \quad v(Rx,t) = Rv(x,t), \quad q(Rx,t) = q(x,t). \tag{3.70}$$

For every $k \in N$ we have the estimate

$$\|\eta(0)\|_{H^k} + \|v(0)\|_{H^k} + \|q(0)\|_{H^k} \le \overline{C}_k\left(\int_{R^2}(1+|\xi|^2)^{k+1}|f(|\xi|)|^2\mathrm{d}\xi\right)^{\frac{1}{2}} < \infty, \tag{3.71}$$

for a constantn $\overline{C}$ depending on the parameters $\rho_0^\pm, p_\pm, g, \sigma, m, l$; moreover, for every $t > 0$ we have $\eta(t), v(t), q(t) \in H^k$ and

$$\begin{cases}\mathrm{e}^{t\lambda_0(f)}\|\eta(0)\|_{H^k} \le \|\eta(t)\|_{H^k} \le \mathrm{e}^{t\Lambda}\|\eta(0)\|_{H^k}, \\ \mathrm{e}^{t\lambda_0(f)}\|v(0)\|_{H^k} \le \|v(t)\|_{H^k} \le \mathrm{e}^{t\Lambda}\|v(0)\|_{H^k}, \\ \mathrm{e}^{t\lambda_0(f)}\|q(0)\|_{H^k} \le \|q(t)\|_{H^k} \le \mathrm{e}^{t\Lambda}\|q(0)\|_{H^k},\end{cases} \tag{3.72}$$

where $\lambda_0(f) = \inf\limits_{|\xi| \in \mathrm{supp}(f)} \lambda(\xi) > 0$ and Λ is given by (3.60).

To state the result, we first define the weighted L^2 norm and the viscosity semi-norm by

$$\|v\|_1^2 = \int_\Omega \rho_0|v|^2, \quad \|v\|_2^2 = \int_\Omega \frac{\varepsilon_0}{2}\left|Dv + Dv^{\mathrm{T}} - \frac{2}{3}(\mathrm{div}v)I\right|^2 + \delta_0|\mathrm{div}v|^2. \tag{3.73}$$

Theorem 3.4 *Let v, η, q be a solution to* (3.22),(3.23) *along with the corresponding jump and boundary conditions. Then in the nonperiodic case*

$$\|v(t)\|_1^2+\|v\|_2^2+\|\partial_t v(t)\|_2^2 \le C\mathrm{e}^{2\Lambda t}\Big(\|\partial_t v(0)\|_1^2+\|v(0)\|_1^2+\|v(0)\|_2^2+\sigma\int_{R^2}|\nabla_{x_1x_2}v_3(0)|^2\Big),$$

for a constant $C = C(\rho_0^{\pm}, p_{\pm}, \Lambda, \varepsilon, \delta, \sigma, g, m, l) > 0$. *In the periodic case with* L *satisfying* (3.57), *the same inequality holds with* Λ *being replaced with* Λ_L.

Theorem 3.5 *In the periodic case let* L *satisfy* (3.58). *For* $j \ge 1$ *define the constants* $K_j \ge 0$ *in terms of the initial data via*

$$K_j = \int_\Omega \rho_0 \frac{|\partial_t^j v(0)|^2}{2} + \int_\Omega \frac{P'(\rho_0)\rho_0}{2}\Big|\mathrm{div}\partial_t^{j-1}v(0) - \frac{g}{p'(\rho_0)}\partial_t^{j-1}v_3(0)\Big|^2 + \int_{(2\pi LT)^2}\frac{\sigma}{2}\big|\nabla_{x_1,x_2}\partial_t^{j-1}v_3(0)\big|^2. \tag{3.74}$$

Then solutions to (3.22),(3.23) *satisfy*

$$\|\eta(t)\|_1 + \|\eta(t)\|_2 \le \|\eta(0)\|_1 + \|\eta(0)\|_2 + t(\|v(0)\|_1 + \|v(0)\|_2) + 2t^{\frac{3}{2}}\sqrt{K_1}, \tag{3.75}$$

$$\|v(t)\|_1 + \|v(t)\|_2 \le \|v(0)\|_1 + \|v(0)\|_2 + 3\sqrt{t}\sqrt{K_1}, \tag{3.76}$$

and for $j \ge 1$

$$\sup_{t\ge0}\frac{1}{2}\|\partial_t^j v(t)\|_1^2 + \int_0^\infty \|\partial_t^j v(t)\|_2^2 \mathrm{d}t \le 2K_j, \tag{3.77}$$

$$\sup_{t\ge0}\|\partial_t^j v(t)\|_2^2 \le \|\partial_t^j v(0)\|_2^2 + 2\sqrt{K_j}\sqrt{K_{j+1}}. \tag{3.78}$$

3.4 A family of modified variational problems

In order to understand λ in a variational framework we consider the two energies

$$E(\varphi,\psi) = \frac{\sigma}{2}|\xi|^2(\psi(0))^2 + \frac{1}{2}\int_{-m}^{l}(\widetilde{\delta} + p'(\rho_0)\rho_0)(\psi' + |\xi|\varphi)^2 - 2g\rho_0|\xi|\varphi\psi + \frac{1}{2}\int_{-m}^{l}\widetilde{\varepsilon}\Big((\varphi' - |\xi|\psi)^2 + (\psi' - |\xi|\varphi)^2 + \frac{1}{3}(\psi' + |\xi|\varphi)^2\Big), \tag{3.79}$$

and

$$J(\varphi,\psi) = \frac{1}{2}\int_{-m}^{l}\rho_0(\varphi^2 + \psi^2), \tag{3.80}$$

which are both well defined on the space $H_0^1((-m,l)) \times H_0^1((-m,l))$. Consider a set

$$\mathcal{A} = \big\{(\varphi,\psi) \in H_0^1((-m,l)) \times H_0^1((-m,l)) | J(\varphi,\psi) = 1\big\}. \tag{3.81}$$

We want to show that the infimum $\inf\limits_{\varphi,\psi\in\mathcal{A}} E(\varphi,\psi)$ solves (3.49),(3.50) along with the corresponding jump and boundary conditions. Also notice that by employing the identity $-2ab = (a-b)^2 - (a^2+b^2)$, and the constraint on $J(\varphi,\psi)$, we may rewrite

$$E(\varphi,\psi) = -g|\xi| + \frac{\sigma|\xi|^2}{2}(\psi(0))^2 + \frac{1}{2}\int_{-m}^{l}(\widetilde{\delta} + p'(\rho_0)\rho_0)(\psi' + |\xi|\varphi)^2 + g|\xi|\rho_0(\varphi-\psi)^2 + \frac{1}{2}\int_{-m}^{l}\widetilde{\varepsilon}\Big((\varphi' - |\xi|\psi)^2 + (\psi' - |\xi|\varphi)^2 + \frac{1}{3}(\psi' + |\xi|\varphi)^2\Big) \ge -g|\xi|, \tag{3.82}$$

where $\widetilde{\varepsilon} = s\varepsilon(\rho_0),\ s \in (0, \infty)$,

$$E(\varphi, \psi) = E(\varphi, \psi; s), \tag{3.83}$$

$$\mu(s) = \inf_{(\varphi,\psi)} E(\varphi, \psi; s). \tag{3.84}$$

Proposition 3.1 *E achieves its infimum on $\mathcal{A}$.*

Proof First note that (3.82) shows that E is bounded below on $\mathcal{A}$. Let $(\varphi_n, \psi_n) \in \mathcal{A}$ be a minimizing sequence. Then φ_n, ψ_n are bounded in $H_0^1((-m, l))$ and $\psi_n(0)$ is bounded in R, so we can choose a subsequence $(\varphi_n, \psi_n) \rightharpoonup (\varphi, \psi)$ weakly in $H_0^1 \times H_0^1$, and $(\varphi_n, \psi_n) \to (\varphi, \psi)$ strongly in $L^2 \times L^2$. The compact embedding $H_0^1 \subset\subset H^{\frac{2}{3}} \hookrightarrow C^0$ implies that $\psi_n(0) \to \psi(0)$ as well. Because of the quadratic structure of all the terms in the integrals defining E, the weak lower semicontinuity and strong L^2 convergence imply that

$$E(\varphi, \psi) \leq \liminf_{n\to\infty} E(\varphi_n, \psi_n) = \inf_{\mathcal{A}} E. \tag{3.85}$$

That $(\varphi, \psi) \in \mathcal{A}$ follows from the strong L^2 convergence.

We now show that the minimizer constructed in the previous result satisfies Euler-Langrange equations equivalent to (3.49),(3.50).

Proposition 3.2 *Let $(\varphi, \psi) \in \mathcal{A}$ be the minimizers of E and $\mu = E(\varphi, \psi)$, then (φ, ψ) are smooth when restricted to $(-m, 0)$ or $(0, l)$ and satisfy*

$$\begin{aligned}\mu\rho_0\varphi = &-(\overline{\varepsilon}\varphi')' + |\xi|^2(4\overline{\varepsilon}/3 + \overline{\delta} + p'(\rho_0)\rho_0)\varphi \\ &+|\xi|\big[(\overline{\delta} + \overline{\varepsilon}/3 + p'(\rho_0)\rho_0)\psi' + (\overline{\varepsilon}' - g\rho_0)\psi\big],\end{aligned} \tag{3.86}$$

$$\begin{aligned}\mu\rho_0\psi = &-\big[(4\overline{\varepsilon}/3 + \overline{\delta} + p'(\rho_0)\rho_0)\psi'\big]' + \overline{\varepsilon}|\xi|^2\psi \\ &-|\xi|\big[((\overline{\delta} + \overline{\varepsilon}/3 + p'(\rho_0)\rho_0)\varphi)' + (g\rho_0 - \overline{\varepsilon'})\varphi\big],\end{aligned} \tag{3.87}$$

along with the jump conditions

$$[[\varphi]] = [[\psi]] = [[\xi(\varphi' - |\xi|\psi)]] = 0, \tag{3.88}$$

$$[[(\overline{\delta} + \overline{\varepsilon}/3 + p'(\rho_0)\rho_0)(\psi' + |\xi|\varphi)]] + [[\overline{\varepsilon}(\psi' - |\xi|\varphi)]] = \sigma|\xi|^2\psi(0), \tag{3.89}$$

and the boundary conditions $\varphi(-m) = \varphi(l) = \psi(-m) = \psi(l) = 0$.

Proof Fix $(\varphi_0, \psi_0) \in H_0^1((-m, l)) \times H_0^1((-m, l))$. Define

$$j(t, \tau) = J(\varphi + t\varphi_0 + \tau\varphi, \psi + t\psi_0 + \tau\psi), \tag{3.90}$$

and note that $j(0, 0) = 1$. Moreover, j is smooth,

$$\frac{\partial j}{\partial t}(0, 0) = \int_{-m}^{l} \rho_0(\varphi_0\psi + \psi_0\psi), \quad \frac{\partial j}{\partial \tau}(0, 0) = \int_{-m}^{l} \rho_0(\varphi^2 + \psi^2) = 2. \tag{3.91}$$

So, by the inverse function theorem, we can solve $\tau = \tau(t)$ in a neighborhood of 0 as a C^1 function of t so that $\tau(0) = 0$, $j(t, z(t)) = 1$. We differentiate the last equation to obtain

$$\frac{\partial j}{\partial t}(0,0)+\frac{\partial j}{\partial \tau}(0,0)\tau'(0)=0, \tag{3.92}$$

hence

$$\tau'(0)=-\frac{1}{2}\frac{\partial j}{\partial t}(0,0)=-\frac{1}{2}\int_{-m}^{l}\rho_0(\varphi_0\varphi+\psi_0\psi). \tag{3.93}$$

Since (φ,ψ) are minimizers over $\mathcal{A}$, we make variations with respect to (φ_0,ψ_0) to find that

$$0=\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}E(\varphi+t\varphi_0+\tau(t)\varphi,\psi+t\psi_0+\tau(t)\psi), \tag{3.94}$$

which implies that

$$\begin{aligned}0=\;&\sigma|\xi|^2\psi(0)\big(\psi_0(0)+\tau'(0)\psi(0)\big)\\&+\int_{-m}^{l}\Big(\overline{\delta}+\frac{\overline{\varepsilon}}{3}+p'(\rho_0)\rho_0\Big)(\psi'+|\xi|\varphi)(\psi_0'+\tau'(0)\psi'+|\xi|\varphi_0+|\xi|\tau'(0)\varphi)\\&-\int_{-m}^{l}g|\xi|\rho_0\big(\psi(\rho_0+\tau'(0)\varphi)+\varphi(\psi_0+\tau'(0)\psi)\big)\\&+\int_{-m}^{l}\overline{\varepsilon}(\psi'-|\xi|\varphi)\big(\psi_0'+\tau'(0)\psi'-|\xi|\varphi_0-|\xi|\tau'(0)\varphi\big)\\&+\int_{-m}^{l}\overline{\varepsilon}(\psi'-|\xi|\varphi)\big(\varphi_0'+\tau'(0)\varphi'-|\xi|\psi_0-|\xi|\tau'(0)\psi\big).\end{aligned} \tag{3.95}$$

Rearranging and plugging in the value of $\tau'(0)$, we rewrite this equation as

$$\begin{aligned}&\sigma|\xi|^2\psi(0)\psi_0(0)+\int_{-m}^{l}\Big(\overline{\delta}+\frac{\overline{\varepsilon}}{3}+p'(\rho_0)\rho_0\Big)(\psi'+|\xi|\varphi)(\psi_0'-|\xi|\varphi_0)-g|\xi|\rho_0(\psi\rho_0+\varphi\psi_0)\\&+\int_{-m}^{l}\overline{\varepsilon}\big((\psi'-|\xi|\varphi)(\psi_0'-|\xi|\varphi_0)+(\varphi'-|\xi|\psi)(\varphi_0'+|\xi|\psi_0)\big)=\mu\int_{-m}^{l}\rho_0(\varphi_0\varphi+\psi_0\psi),\end{aligned} \tag{3.96}$$

where the Lagrange multiplier is $\mu=E(\varphi,\psi)$. Since φ_0,ψ_0 are independent, this gives rise to the pair of equations

$$\begin{aligned}&\int_{-m}^{l}\overline{\varepsilon}\varphi'\varphi_0'+(\overline{\delta}+4\overline{\varepsilon}/3+p'(\rho_0)\rho_0)|\xi|^2\varphi\varphi_0-\int_{-m}^{l}(\overline{\varepsilon}|\xi|\psi p_0)'\\&+|\xi|\int_{-m}^{l}[(\overline{\delta}+\overline{\varepsilon}/3+p'(\rho_0)\rho_0)\psi'+(\overline{\varepsilon}'-g\rho_0)\psi]\varphi_0=\mu\int_{-m}^{l}\rho_0\varphi\varphi_0,\end{aligned} \tag{3.97}$$

and

$$\begin{aligned}&\sigma|\xi|^2\psi(0)\psi_0(0)+\int_{-m}^{l}\Big[\Big(\overline{\delta}+\frac{4}{3}\overline{\varepsilon}+p'(\rho_0)\rho_0\Big)\psi'+\Big(\overline{\delta}+\frac{\overline{\varepsilon}}{3}+p'(\rho_0)\rho_0\Big)|\xi|\varphi\Big]\psi'\\&-\int_{-m}^{l}(\overline{\varepsilon}|\xi|\varphi\psi_0)'+\int_{-m}^{l}\left[\overline{\varepsilon}|\xi|^2\psi+(\overline{\varepsilon}-g\rho_0)|\xi|\varphi\right]\psi_0=\mu\int_{-m}^{l}\rho_0\psi\psi_0.\end{aligned} \tag{3.98}$$

By making variations with φ_0,ψ_0 compactly supported in either $(-m,0)$ or $(0,l)$, we find that φ,ψ satisfy (3.86),(3.87) in a weak sense in $(-m,0)$ or $(0,l)$. The standard bootstrapping arguments then show that (φ,ψ) are in $H^k(-m,0)(H^k(0,l))$ for any $k\geq 0$, and hence the functions are smooth when restricted to either interval. To show that the jump conditions are satisfied, integrating the terms in (3.97), we find that

$$[[\overline{\varepsilon}(\varphi'-|\xi|\psi)]]\varphi(0)=0. \tag{3.99}$$

Performing a similar integration by parts in (3.98) yields the jump condition

$$0=\sigma|\xi|^2\psi(0)-[[(\overline{\delta}+\overline{\varepsilon}/3+p'(\rho_0)\rho_0)+\psi'+|\xi|\varphi]]-[[\overline{\varepsilon}(\psi'-|\xi|\varphi)]]. \tag{3.100}$$

The conditions $[[\varphi]]=[[\psi]]=0,\ \varphi(-m)=\varphi(l)=\psi(-m)=\psi(l)=0$ are satisfied trivially since

$$\varphi,\psi\in H_0^1(-m,l)\hookrightarrow C_0^{0,\frac{1}{2}}((-m,l)).$$

We now show that for s sufficiently small, the infimum $\inf E$ over $\mathcal{A}$ is in fact negative.

Proposition 3.3 *Suppose that* $0<|\xi|^2<g[[\rho_0]]/\sigma$. *Then there exists an* $s_0>0$ *depending on the quantities* $\rho_0^{\pm}$, $p_{\pm}$, g, $\varepsilon_{\pm}$, σ, $m,l,|\xi|$ *so that for* $s\leq s_0$, *there is* $\mu(s)<0$.

Proof Since both E and J are homogeneous of degree 2, it suffices to show that

$$\inf_{(\varphi,\psi)\in H_0^1\times H_0^1}\frac{E(\varphi,\psi)}{J(\varphi,\psi)}<0, \tag{3.101}$$

but since J is positive definite, we may reduce to construct any pair$(\varphi,\psi)\in H_0^1\times H_0^1$ such that $E(\varphi,\psi)<0$. We assume that $\varphi=-\psi'/|\xi|$ so that the first integrand term in $E(\varphi,\psi)$ is vanished. We must then construct a $\psi\in H_0^2$ so that

$$\overline{E}(\psi)=E\Big(-\frac{\psi'}{|\xi|},\psi\Big)=\frac{\sigma|\xi|^2}{2}(\psi(0))^2+\int_{-m}^{l}g\rho_0\psi\psi'+\frac{\overline{\varepsilon}}{2}\left(\Big(\frac{\psi'''}{|\xi|}+|\xi|\psi\Big)^2+4(\psi')^2\right)<0.$$

We employ the identity $\psi\psi'=((\psi)^2)'/2$, an integration by parts, and the fact that ρ_0 solves

$$\frac{\mathrm{d}p(\rho_0)}{\mathrm{d}x_3}=-g\rho_0$$

to write

$$\begin{aligned}\int_{-m}^{l}g\rho_0\psi\psi'&=\left[\frac{g\rho_0\psi^2}{2}\right]_0^l-\frac{1}{2}\int_0^l g\rho_0'\psi^2+\left[\frac{g\rho_0\psi^2}{2}\right]_{-m}^0-\frac{1}{2}\int_{-m}^0 g\rho_0'\psi^2\\&=-\frac{g[\psi(0)]^2}{2}[[\rho_0]]+\frac{g^2}{2}\int_{-m}^{l}\frac{\rho_0}{p'(\rho_0)}\psi^2.\end{aligned} \tag{3.102}$$

Note that $[[\rho_0]]=\rho^+-\rho^->0$ so that the right-hand side is not positive definite.

For $\alpha>5$ we define the test function $\psi_\alpha\in H_0^2(-m,l)$ according to

$$\psi_\alpha(x_3)=\begin{cases}c\Big(1-\dfrac{x_3^2}{l^2}\Big)^{\frac{\alpha}{2}}, & x_3\in(0,l),\\ \Big(1+\dfrac{x_3^2}{m^2}\Big)^{\frac{\alpha}{2}}, & x_3\in(-m,0).\end{cases} \tag{3.103}$$

Simple calculations then show that

$$\int_{-m}^{l}(\psi_\alpha)^2=\frac{\sqrt{\pi}(m+l)\Gamma(\alpha+1)}{2\Gamma(\alpha+\frac{3}{2})}=o_\alpha(1), \tag{3.104}$$

where $o_\alpha(1)$ is a quantity that vanishes as $\alpha\to\infty$, and that

$$\int_{-m}^{l}\Big(\Big(\frac{\psi''}{|\xi|}+|\xi|\psi\Big)^2+4(\psi')^2\Big)\le C, \tag{3.105}$$

for a constant C depending on $\alpha,m,l,|\xi|$. Combining these, we find that

$$\widetilde{E}(\psi_\alpha)\le\frac{\sigma|\xi|^2-g[[\rho_0]]}{2}+o_\alpha(1)+C, \tag{3.106}$$

for a constant C depending on $\alpha,\rho_0^\pm,p_\pm,g,\varepsilon_\pm,m,l,|\xi|$. Since $\sigma|\xi|^2<g[[\rho_0]]$ we can choose an α sufficiently large so that the sum of the first two terms is strictly negative. Then there exists an $s_0>0$ depending on the various parameters so that for $s\le s_0$, we have $\widetilde{E}(\psi_\alpha)<0$, thereby this proves the result.

Remark 3.2 We can prove $\mu(s)\le -C_0+sC_1$ by

$$\frac{\sigma|\xi|^2-g[[\rho_0]]}{2}(\psi(0))^2<0$$

$\psi(0)\neq0,\ |\xi|^2<g[[\rho_0]]/\sigma,\ E(\varphi,\psi)<0$.

Lemma 3.1 *Suppose that* $(\varphi,\psi)\in A$ *satisfies* $E(\varphi,\psi)<0$. *Then* $\psi(0)\neq 0,\ |\xi|^2<g[[\rho_0]]/\sigma$.

Proof A completion of the square allows us to write

$$\begin{aligned}&p'(\rho_0)\rho_0(\psi'+|\xi|\varphi)^2-2g\rho_0|\xi|\psi\varphi\\ =&\Big(\sqrt{p'(\rho_0)\rho_0}(\psi'+|\xi|\varphi)-\frac{g\sqrt{\rho_0}}{\sqrt{p'(\rho_0)}}\psi\Big)^2+2g\rho_0\psi\psi'-\frac{g^2\varphi_0}{p'(\rho_0)}\psi^2.\end{aligned} \tag{3.107}$$

Integrating by parts as in (3.102), we know that

$$\int_{-m}^{l}2g\rho_0\psi\psi'-\frac{g^2\varphi_0}{p'(\rho_0)}\psi^2=-g[[\rho_0]](\psi(0))^2. \tag{3.108}$$

Combining these equalities, we can rewrite $E(\varphi,\psi)$ as

$$\begin{aligned}E(\varphi,\psi)=&\frac{1}{2}\int_{-m}^{l}\Big(\overline{\delta}+\frac{\overline{\varepsilon}}{3}\Big)(\psi'+|\xi|\varphi)^2+p'(\rho_0)\rho_0\Big((\psi'+|\xi|\varphi)-\frac{g}{p'(\rho_0)}\psi\Big)^2\\ &+\frac{1}{2}\int_{-m}^{l}\overline{\varepsilon}((\varphi'-|\xi|\psi)^2+(\psi'-|\xi|\varphi)^2)+\frac{\sigma|\xi|^2-g[[\rho_0]]}{2}\psi^2(0).\end{aligned}$$

From the nonnegativity of the integrals, if $E(\varphi,\psi)<0$, then $\psi(0)\neq 0$, $|\xi|^2<g[[\rho_0]]/\sigma$.

Proposition 3.4 *Let* $\mu:(0,\infty)\to R$ *be given by* (3.84), *then the following conclusions hold:*

1. $\mu\in C^{0,1}_{loc}((0,\infty))$, *and in particular* $\mu\in C^0((0,\infty))$.
2. *There exists a positive constant* $C_2=C_2(\rho_0^{\pm},\rho_{\pm},g,\varepsilon_{\pm},\sigma,m,l)$ *so that*

$$\mu(s)\geq -g|\xi|+sC_2. \tag{3.109}$$

3. $\mu(s)$ *is strictly increasing.*

Proof Fix a compact interval $Q=[a,b]\subset C(0,\infty)$ and any pair $(\varphi_0,\psi_0)\in A$. We may decompose E according to

$$E(\varphi,\psi,s)=E_0(\varphi,\psi)+sE_1(\varphi,\psi), \tag{3.110}$$

for

$$E_0(\varphi,\psi)=\frac{\sigma|\xi|^2}{2}(\psi(0))^2+\frac{1}{2}\int_{-m}^{l}p'(\rho_0)\rho_0(\psi'+|\xi|\varphi)^2-2g\rho_0|\xi|\psi\varphi \tag{3.111}$$

and

$$E_1(\varphi,\psi)=\frac{1}{2}\int_{-m}^{l}\Big(\delta_0+\frac{\varepsilon_0}{3}\Big)(\psi'+|\xi|\varphi)^2+\varepsilon_0((\psi'-|\xi|\varphi)^2+(\psi'+|\xi|\varphi)^2)\geq 0. \tag{3.112}$$

The nonnegativity of E_1 implies that E is nondecreasing in s with a fixed $(\varphi,\psi)\in A$.

Now, by Proposition 3.1, for each $s\in(0,+\infty)$ we can find a pair $(\varphi,\psi)\in A$ so that

$$E(\varphi_s,\psi_s;s)=\inf_{(\varphi,\psi)\in A}E(\varphi,\psi;s)=\mu(s). \tag{3.113}$$

We deduce from the nonnegativity of E_1, the minimality of (φ_s,ψ_s), and the equality (3.82) that

$$E(\varphi_0,\psi_0;b)\geq E(\varphi_0,\psi_0;s)\geq sE_1(\varphi_s,\psi_s)-g|\xi|, \tag{3.114}$$

for all $s\in Q$. This implies that there exists a constant $0<K=K(a,b,\varphi_0,\psi_0,g,|\xi|)<\infty$ so that

$$\sup_{s\in Q}E_1(\varphi_s,\psi_s)<K. \tag{3.115}$$

Let $s\in Q$, $i=1,2$. Using the minimality of $(\varphi_{s_1},\psi_{s_1})$ compared to $(\varphi_{s_2},\psi_{s_2})$ we know that

$$\mu(s_1)=E(\varphi_{s_1},\psi_{s_1};s_1)\leq E(\varphi_{s_2},\psi_{s_2};s_1), \tag{3.116}$$

but from the decomposition (3.110), we obtain

$$\begin{aligned}E(\varphi_{s_2},\psi_{s_2};s_1)&\leq E(\varphi_{s_2},\psi_{s_2};s_2)+|s_1-s_2|E_1(\varphi_{s_2},\psi_{s_2})\\&=\mu(s_2)+|s_1-s_2|E_1(\varphi_{s_2},\psi_{s_0}).\end{aligned} \tag{3.117}$$

Together with these two inequalities and employing (3.115), we find that

$$\mu(s_1) \leq \mu(s_2) + K|s_1 - s_2|. \tag{3.118}$$

Similarly we can obtain $\mu(s_2) \leq \mu(s_1) + K|s_1 - s_2|$. Thus

$$|\mu(s_1) - \mu(s_2)| \leq K|s_1 - s_2|, \tag{3.119}$$

which proves the first assertion.

To prove (3.109) we note that equality (3.82) and the nonnegativity of E_1 imply that

$$\mu(s) \geq -g|\xi| + s \inf_{(\varphi,\psi)\in\mathcal{A}} E_1(\varphi, \psi). \tag{3.120}$$

We can easily verify that this infimum, denoted by a constant C_2, is positive.

Finally, to prove the third assertion, note that if $0 < s_1 < s_2 < \infty$, then the decomposition (3.110) implies that

$$\mu(s_1) = E(\varphi_{s_1}, \psi_{s_1}; s_1) \leq E(\varphi_{s_2}, \psi_{s_2}; s_1) \leq E(\varphi_{s_2}, \psi_{s_2}; s_2) = \mu(s_2). \tag{3.121}$$

This shows that μ is nondecreasing in s. Now suppose by way of contradiction that $\mu(s_1) = \mu(s_2)$. Then the previous inequality implies that

$$s_1 E_(\varphi_{s_2}, \psi_{s_2}) = s_2 E_(\varphi_{s_2}, \psi_{s_2}), \tag{3.122}$$

which means that $E_1(\varphi_{s_2}, \psi_{s_2}) = 0$. This in turn forces $\varphi_{s_2} = \psi_{s_2} = 0$, which contradicts the fact that $(\varphi_{s_2}, \psi_{s_2}) \in \mathcal{A}$. Hence the equality cannot be achieved, and μ is strictly increasing in s.

Now we know that when $0 < |\xi|^2 < g[[\rho_0]]/\gamma$, the eigenvalue $\mu(s)$ is a continuous function. We can then define an open set $S = \mu^{-1}((-\infty, 0)) \subset (0, \infty)$, on which we can calculate $\lambda = \sqrt{-\mu}$. Note that S is nonempty by Proposition 3.3.

We can now state a result about the existence of solutions to (3.49),(3.50) for these values of $|\xi|, s$. To emphasize the dependence on the parameters, we write

$$\varphi = \varphi_s(|\xi|, x_3), \quad \psi = \psi_s(|\xi|, x_3), \quad \lambda = \lambda(|\xi|, s). \tag{3.123}$$

Proposition 3.5 *For each $s \in S$, $0 < |\xi|^2 < g[[\rho_0]]/\sigma$, there exists a solution $\varphi_s(|\xi|, x_3)$, $\psi_s(|\xi|, x_3)$ with $\lambda = \lambda(|\xi|, s) > 0$ to problem* (3.49),(3.50) *along with the corresponding jump and boundary conditions. Solutions $\psi_s(|\xi|, 0) \neq 0$ are smooth when restricted to either $(-m, 0)$ or $(0, l)$.*

Proof Let $\varphi_s(|\xi|, \cdot), \psi_s(|\xi|, \cdot) \in \mathcal{A}$ be the solutions to (3.86),(3.87). Since $s \in S$, we may write $\mu = -\lambda^2$, $\lambda > 0$, which means that the pair $\varphi_s(|\xi|, s), \psi_s(|\xi|, s)$ solve problem (3.49),(3.50). The fact that $\psi_s(|\xi|, 0) \neq 0$ follows from Lemma 3.1.

In order to obtain solutions to the original problem, we must be able to find an $s \in S$, so that $s = \lambda(|\xi|, s)$. It turns out that the set S is sufficiently large to accomplish this.

Theorem 3.6 *There exists a unique $s \in S$, so that $\lambda(|\xi|, s) = \sqrt{-\mu(s)} > 0$ and*

$$s = \lambda(|\xi|, s). \tag{3.124}$$

Proof According to Remark 3.2, we know that $\mu(s) \le -C_0 + sC_1$. Moreover, the lower bound (3.109) implies that $\mu(s) \to +\infty$ as $s \to \infty$. Since μ is continuous and strictly increasing, there exists an $s \in (0, \infty)$ so that $S = \mu^{-1}((-\infty, 0)) = (0, s_*)$. Since $\mu > 0$ on S, we define $\lambda = \sqrt{-\mu}$. Now define a function $\Phi : (0, s_*) \to (0, \infty)$ according to $\Phi(s) = s/\lambda(|\xi|, s)$. It can be easily check that Φ is continuous and strictly increasing in s. Also, $\lim\limits_{s\to 0} \Phi(s) = 0$, $\lim\limits_{s\to s_*} \Phi(s) = +\infty$. Then by the intermediate value theorem, there exists an $s \in (0, s_*)$ so that $\Phi(s) = 1$, that is $s = \lambda(|\xi|, s)$. This is unique since Φ is strictly increasing.

We now use Theorem 3.6 to study $s = s(|\xi|)$. Since for each fixed $0 < |\xi|^2 < g[[\rho_0]]/\sigma$, we can uniquely find an $s \in S$ so that (3.124) holds. we also write $\lambda = \lambda(|\xi|)$. Using this new notation and the solutions to the equations given by Proposition 3.5, we can construct solutions to system (3.49),(3.50) as well.

Proof of Theorem 3.1 We may find a rotation operator $R \in SO(2)$, so that $R\xi = (|\xi|, 0)$. For $s = s(\xi)$ given by Theorem 3.6, define $(\varphi(\xi, x_3), \theta(\xi, x_3)) = R^{-1}(\varphi_s(|\xi|, x_3), 0)$ and $\psi(\xi, x_3) = \psi_s(|\xi|, x_3)$, where the functions $\varphi_s(|\xi|, x_3), \psi_s(|\xi|, x_3)$ are the solutions from Proposition 3.5. This gives a solution to (3.31)-(3.33). The equivariance in ξ follows from the definition.

3.5 Solutions to (3.22),(3.23)

In this section we will construct growing solutions to (3.22),(3.23) using the solutions to (3.31)-(3.33) constructed in Theorem 3.1. In the periodic case this can only be done when L satisfies (2.17), but the construction is essentially trivial since normal mode solutions are in $L^2(\Omega)$. In the nonperiodic case, we must resort to a Fourier synthesis of the normal modes in order to produce L^2 solutions.

We begin by defining some terms. For a function $f \in L^2(\Omega)$, we define the horizontal Fourier transform in the nonperiodic case via

$$\widehat{f}(\xi_1, \xi_2, x_3) = \int_{R^2} f(x_1, x_2, x_3) \mathrm{e}^{-i(x_1\xi_1 + x_2\xi_2)} \mathrm{d}x_1 \mathrm{d}x_2, \tag{3.125}$$

for $\xi \in R^2$. In the periodic case the integral over R^2 must be replaced with an integral over $(2\pi L_T)^2$ for $\xi \in (L^{-1}Z)^2$. In the nonperiodic case, by the Fubini and Parseval theorems, we have $\widehat{f} \in L^2(\Omega)$ and

$$\int_\Omega |f(x)|^2 \mathrm{d}x = \frac{1}{4\pi^2} \int_\Omega |\widehat{f}(\xi, x_3)|^2 \mathrm{d}\xi \mathrm{d}x_3. \tag{3.126}$$

The period is replaced with $4\pi L^2$ and the integral is replaced with a sum over $(L^{-1}Z)^2$ on the right-hand side.

We now define the piecewise *Sobolev* spaces. For a function f defined on Ω we write f_+ for the restriction to Ω_+ and f_- for the restriction to Ω_-. For $k \in N$, define the piecewise Sobolev space of order k by

$$H^k(\Omega) = \{f|\, f_+ \in H^k(\Omega_+), f_- \in H^k(\Omega_-)\}, \tag{3.127}$$

endowed with the norm $\|f\|^2_{H^k} = \|f\|^2_{H^k(\Omega_+)} + \|f\|^2_{H^k(\Omega_-)}$. Writing $I_- = (-m, 0)$, $I_+ = (0, l)$, we can take the norms to be given as

$$\|f\|^2_{H^k} = \sum_{j=0}^{k} \int_{R^2} (1 + |\xi|^2)^{k-j} \|\partial^j_{x_3} \widehat{f}_\pm(\xi, \cdot)\|^2_{L^2(I_\pm)} \mathrm{d}\xi. \tag{3.128}$$

Lemma 3.2 *Suppose $0 < a < b < |\xi|_c$ and that $|\xi| \in [a, b]$. Let (φ, θ, ψ) be the solutions constructed in Theorem* 3.1. *Then for each $k \geq 0$ there exists a constant $A_k > 0$ depending on the parameters $a, b, \rho_0^\pm, P_\pm, g, \varepsilon_\pm, \delta_\pm, m, l$,*

$$\begin{aligned}&\|\varphi(\xi, \cdot)\|_{H^k((-m,0))} + \|\theta(\xi, \cdot)\|_{H^k((-m,0))} + \|\psi(\xi, \cdot)\|_{H^k((-m,0))}\\ &+ \|\varphi(\xi, \cdot)\|_{H^k((0,l))} + \|\theta(\xi, \cdot)\|_{H^k((0,l))} + \|\psi(\xi, \cdot)\|_{H^k((0,l))} \leq A_k.\end{aligned}$$

Also, there exists a $B_0 > 0$ depending on the same parameters so that

$$\|\varphi(\xi, \cdot) + \theta(\xi, \cdot) + \psi(\xi, \cdot)\|_{L^2((-m,l))} \geq B_0.$$

Proof of Theorem 3.2 It is clear that η, v, q defined in this way are solutions to (3.22),(3.23). That they are real-valued follows from the equivariance in ξ stated in Theorem 3.1 The solutions are in $H^k(\Omega)$ at $t = 0$ because of Lemma 3.2. The growth in time stated in (3.65) follows from the definition of η, v, q.

Proof of Theorem 3.3 For each fixed $\xi \in R^2$ so that $|\xi| \in (0, |\xi|_c)$,

$$\eta(x, t) = f(|\xi|)\widehat{w}(\xi, x_3)\mathrm{e}^{\lambda(|\xi|)t}\mathrm{e}^{ix'\cdot\xi}, \tag{3.129}$$

$$v(x, t) = \lambda(|\xi|)f(|\xi|)\widehat{w}(\xi, x_3)\mathrm{e}^{\lambda(|\xi|)t}\mathrm{e}^{ix'\cdot\xi}, \tag{3.130}$$

$$q(x, t) = -\rho_0(x_3)f(|\xi|)(\xi_1\varphi(\xi, x_3) + \xi_2\theta(\xi, x_3) + \partial_3\psi(\xi, x_3))\mathrm{e}^{ix'\cdot\xi}, \tag{3.131}$$

then we obtain a solution to (3.22),(3.23).

Since $\mathrm{supp}(f) \subset (0, |\xi|_c)$, Lemma 3.2 implies that

$$\sup_{\xi\in\mathrm{supp}(f)} \|\partial^k_{x_3}\widehat{w}(\xi, \cdot)\|_{L^\infty} \leq \infty, \quad \text{for any } k \in N. \tag{3.132}$$

This together with the definition of Λ, and the dominated convergence theorem implies that the Fourier synthesis of these solutions given by (3.67)-(3.69) is also a solution that is smooth when restricted to $\Omega_\pm$. The Fourier synthesis is real-valued because $f(|\xi|)$ is real-valued and radial because of the equivariance in ξ given in Theorem 3.1. This equivariance in ξ also implies the equivariance of η, v, q written in (3.70).

The bound (3.71) follows by applying Lemma 3.2 with arbitrary $k \geq 0$ and utilizing the fact that f is compactly supported. The compact support of f also implies that $\lambda_0(f) > 0$, so that $\lambda_0(f) \leq \lambda(|\xi|) \leq \Lambda$, $|\xi| \in \mathrm{supp}(f)$. This then yields the bounds (3.72).

3.6 Growth of solutions to the linearized problem

In this section we will prove estimates for the growth in time of arbitrary solutions to (3.22),(3.23) in terms of the largest growing mode: Λ in the nonperiodic case and Λ_L in the periodic case, defined by (3.60) and (3.61) respectively. To this end, we suppose that η, v, q are real-valued solutions to (3.22),(3.23) along with the corresponding jump and boundary conditions.

It will be convenient to work with a second-order formulation of the equations. To arrive at this, we differentiate the third equation in time and eliminate the q, η terms using the other equations. This yields the equation

$$\begin{aligned}&\rho_0\partial_{tt}v - \nabla(P'(\rho_0)\rho_0\mathrm{div}v) + g\rho_0\nabla v_3 - g\rho_0\mathrm{div}ve_3\\&= \mathrm{div}\Big(\varepsilon_0(D\partial_t v + D\partial_t v^{\mathrm{T}}) - \frac{2}{3}(\mathrm{div}\partial_t v)I + \delta_0(\mathrm{div}\partial_t v)I\Big),\end{aligned} \tag{3.133}$$

coupled to the jump conditions $[[\partial_t v]] = 0$, and

$$\Big[\Big[(p'(\rho_0)\rho_0\mathrm{div}v)I + \varepsilon_0(D\partial_t v + D\partial_t v^{\mathrm{T}}) + \Big(\delta_0 - \frac{2}{3}\varepsilon_0\Big)\mathrm{div}\partial_t vI\Big]\Big]e_3 = -\sigma\triangle_{x_1,x_2}v_3 e_3. \tag{3.134}$$

The function $\partial_t v$ also satisfies $\partial_t v(x_1, x_2, -m, t) = \partial_t v(x_1, x_2, l, t) = 0$ at the upper and lower boundaries. The initial data for $\partial_t v(x_1, x_2, -m, t) = \partial_t v(x_1, x_2, l, t) = 0$ is given in terms of the initial data $q(0), v(0)$ and $\eta(0)$ via the third linear equation; that is, $\partial_t v(0)$ satisfies

$$\begin{aligned}\rho_0\partial_t v(0) &= -g\rho(0)e_3 - g\rho_0\nabla\eta_3(0)\\&\quad+\mathrm{div}\Big(\varepsilon_0(Dv(0) + Dv(0)^{\mathrm{T}}) - \frac{2}{3}(\mathrm{div}(v(0))I) + \delta_0(\mathrm{div}(v(0)))I\Big).\end{aligned} \tag{3.135}$$

The following lemma gives an energy and its evolution equation for solutions to the second-order problem.

Lemma 3.3 *Let v solve* (3.133) *with the corresponding jump and boundary conditions. Then in the nonperiodic case,*

$$\begin{aligned}&\partial_t\int_\Omega \rho_0\frac{|\partial_t v|^2}{2} + \frac{p'(\rho_0)\rho_0}{2}\Big|\mathrm{div}v - \frac{g}{p'(\rho_0)}v_3\Big|^2 + \int_\Omega\frac{\varepsilon_0}{2}\Big|D\partial_t v + D\partial_t v^{\mathrm{T}} - \frac{2}{3}(\mathrm{div}\partial_t v)I\Big|^2 + \int_\Omega\delta_0|\mathrm{div}\partial_t v|^2\\&= \partial_t\int_\Omega\frac{g[[\rho_0]]}{2}|v_3|^2 - \frac{\sigma}{2}|\nabla_{x_1,x_2}v_3|^2.\end{aligned} \tag{3.136}$$

In the periodic case, the same equation holds by replacing the integral over R^2 with an integral over $(2\pi L\mathbb{T})^2$.

Proof We will prove the result in the nonperiodic case, Since the periodic case follows similarly. Take the dot product of (3.133) with v_t and integrate over Ω_t. After integrating by parts, we get

$$\begin{aligned}&\int_{\Omega_+}\rho_0\partial_t v\cdot\partial_{tt}v+p'(\rho_0)\rho_0(\mathrm{div}v)(\mathrm{div}\partial_t v)-g\rho_0(v_3\mathrm{div}\partial_t v+\partial_t v_3\mathrm{div}v)+\frac{g^3\rho_0}{p'(\rho_0)}v_3\partial_t v_3\\&+\int_{\Omega_+}\frac{\varepsilon_0}{2}\Big|D\partial_t v+D\partial_t v^{\mathrm{T}}-\frac{2}{3}(\mathrm{div}\partial_t v)I\Big|^2+\int_{\Omega_+}\delta_0|\mathrm{div}\partial_t v|^2\\=&\int_{\mathbb{R}^2}g\rho_0^+v_3\partial_t v_3-\int_{\mathbb{R}^2}p'_+(\rho_0^+)\rho_0^+\mathrm{div}v\partial_t v_3-\int Te_3\cdot\partial_t v,\end{aligned}\tag{3.137}$$

where $T=(p'(\rho_0)\rho_0\mathrm{div}v)I+\varepsilon_0\big(D\partial_t v+D\partial_t v^{\mathrm{T}}-2(\mathrm{div}\partial_t v)I/3\big)+\delta_0\mathrm{div}\partial_t vI$.

Taking the derivatives of both sides of the above equation leads to

$$\begin{aligned}&\frac{\partial}{\partial t}\int_{\Omega_+}\rho_0\frac{|\partial_t v|^2}{2}+\frac{p'(\rho_0)\rho_0}{2}\Big|\mathrm{div}v-\frac{g}{p'(\rho_0)}v_3\Big|^2\\&+\int_{\Omega_+}\frac{\varepsilon_0}{2}\Big|D\partial_t v+D\partial_t v^{\mathrm{T}}-\frac{2}{3}(\mathrm{div}\partial_t v)I\Big|^2+\int_{\Omega_+}\delta_0|\mathrm{div}\partial_t v|^2\\=&\frac{\partial}{\partial t}\int_{\mathbb{R}^2}g\rho_0^+\frac{|v_3|^2}{2}-\int_{\mathbb{R}^2}Te_3\cdot\partial_t v.\end{aligned}$$

A similar result holds on $\Omega_-=\mathbb{R}^2\times(-m,0)$ with the opposite sign on the right-hand side. Adding these two equalities yields

$$\begin{aligned}&\frac{\partial}{\partial t}\int_{\Omega}\rho_0\frac{|\partial_t v|^2}{2}+\frac{p'(\rho_0)\rho_0}{2}\Big|\mathrm{div}v-\frac{g}{p'(\rho_0)}v_3\Big|^2\\&+\int_{\Omega}\frac{\varepsilon_0}{2}\Big|D\partial_t v+D\partial_t v^{\mathrm{T}}-\frac{2}{3}(\mathrm{div}\partial_t v)I\Big|^2+\int_{\Omega}\delta_0|\mathrm{div}\partial_t v|^2\\=&\partial_t\int_{\mathbb{R}^2}\frac{g[[\rho_0]]}{2}|v_3|^2-\int_{\mathbb{R}^2}[[Te_3\cdot\partial_t v]].\end{aligned}\tag{3.138}$$

Using the jump conditions, we find that

$$-\int_{\mathbb{R}^2}[[Te_3\cdot\partial_t v]]=\int_{\mathbb{R}^2}\sigma\triangle_{x_1,x_2}v_3\partial_t v_3=-\sigma\int_{\mathbb{R}^2}\nabla_{x_1,x_2}v_3\nabla_{x_1,x_2}\partial_t v_3=-\partial_t\int_{\mathbb{R}^2}\frac{\sigma}{2}|\nabla_{x_1,x_2}v_3|^2.$$

The result follows by substituting this into (3.138).

The following result is devoted to estimating the energy in terms of Λ.

Lemma 3.4 *Let $v\in H^1(\Omega)$ be so that $v(x_1,x_2,-m)=v(x_1,x_2,l)=0$. In the nonperiodic case we have the inequality*

$$\begin{aligned}&\int_{\mathbb{R}^2}\frac{g[[\rho_0]]}{2}|v_3|^2-\frac{\sigma}{2}|\nabla_{x_1,x_2}v_3|^2-\int_{\Omega}\frac{p'(\rho_0)\rho_0}{2}\Big|\mathrm{div}v-\frac{g}{p'(\rho_0)}v_3\Big|^2\\\le&\frac{\Lambda^2}{2}\int_{\Omega}\rho_0|v|^2+\frac{\Lambda^2}{2}\int_{\Omega}\frac{\varepsilon_0}{2}\Big|Dv+Dv^{\mathrm{T}}-\frac{2}{3}(\mathrm{div}v)I\Big|^2+\delta_0|\mathrm{div}v|^2.\end{aligned}\tag{3.139}$$

In the periodic case, if $\sqrt{\sigma/g[[\rho_0]]} < L$*, then the same inequality holds by replacing* $\mathbb{R}^2$ *integral replaced with an integral over* $(2\pi L\mathbb{T})^2$*, and replacing* Λ *with* Λ_L*.*

Proof We also only prove only the nonperiodic version. Take the horizontal Fourier transform and apply (3.126) to see that

$$\begin{aligned}
&4\pi^2\int_{\mathbb{R}^2}\frac{g[[\rho_0]]}{2}|v_3|^2-\frac{\sigma}{2}|\nabla_{x_1,x_2}v_3|^2-4\pi^2\int_\Omega\frac{p'(\rho_0)\rho_0}{2}\left|\mathrm{div}v-\frac{g}{p'(\rho_0)}v_3\right|^2\\
&=\int_{R^2}\frac{g[[\rho_0]]-\sigma|\xi|^2}{2}|\widehat{v}_3|^2-\int_\Omega\frac{p'(\rho_0)}{2}\rho_0\left|i\xi_1\widehat{v}_1+i\xi_2\widehat{v}_2+\partial_3\widehat{v}_3-\frac{g}{p'(\rho_0)}\widehat{v}_3\right|^2\mathrm{d}\xi\mathrm{d}x_3\\
&=\int_{R^2}\Big(\frac{g[[\rho_0]]-\sigma|\xi|^2}{2}|\widehat{v}_3|^2-\int_{-m}^{l}\frac{p'(\rho_0)}{2}\rho_0\left|i\xi_1\widehat{v}_1+i\xi_2\widehat{v}_2+\partial_3\widehat{v}_3-\frac{g}{p'(\rho_0)}\widehat{v}_3\right|^2\mathrm{d}x_3\Big)\mathrm{d}\xi.
\end{aligned}\tag{3.140}$$

Consider now the last integrand for fixed $\xi\neq0$, and let $\varphi(x_3)=i\widehat{v}_1(\xi,x_3)$, $Q(x_3)=i\widehat{v}_2(\xi,x_3)$, $\psi(x_3)=\widehat{v}_3(\xi,x_3)$. That is, define

$$Z(\varphi,\theta,\psi;\xi)=\frac{g[[\rho_0]]-\sigma|\xi|^2}{2}|\psi|^2-\int_{-m}^{l}\frac{p'(\rho_0)\rho_0}{2}\left|\xi_1\varphi+\xi_2\theta-\frac{g}{p'(\rho_0)}\psi\right|^2\mathrm{d}x_3.\tag{3.141}$$

Denoting $Z(\varphi,\theta,\psi;\xi)=Z(\Re\varphi,\Re\theta,\Re\psi;\Re\xi)+Z(\Im\varphi,\Im\theta,\Im\psi;\Im\xi)$, Z is bounded when φ,θ,ψ are real-valued functions, which can be applied to study the real and imaginary parts of φ,θ,ψ.

The expression for Z is invariant under simultaneous rotations of ξ and (φ,θ), so without loss of generality we may assume that $\xi=(|\xi|,0)$, $|\xi|>0$ and $\theta=0$. If $\sigma>0$, then we assume that $|\xi|<|\xi|_c$. Using (3.79) with $\widetilde{\varepsilon}=\lambda(|\xi|)\varepsilon_0$, $\widetilde{\delta}=\lambda(|\xi|)\delta_0$, we may rewrite Z as

$$\begin{aligned}
Z(\varphi,\theta,\psi;\xi)=&-E(\varphi,\psi;\lambda(|\xi|))+\frac{\lambda(|\xi|)}{2}\int_{-m}^{l}\delta_0\big|\psi'+|\xi|\varphi\big|^2\\
&+\frac{\lambda(|\xi|)}{2}\int_{-m}^{l}\varepsilon_0\Big(\big|\varphi'-|\xi|\varphi\big|^2+\big|\psi'-|\xi|\varphi\big|^2+\frac{1}{3}\big|\psi'+|\xi|\varphi\big|^2\Big),
\end{aligned}\tag{3.142}$$

hence

$$\begin{aligned}
Z(\varphi,\theta,\psi;\xi)\le&\frac{\Lambda^2}{2}\int_{-m}^{l}\rho_0(|\varphi|^2+|\psi|^2)+\frac{\Lambda}{2}\int_{m}^{l}\delta\big|\psi'+|\xi|\varphi\big|^2\\
&+\frac{\Lambda}{2}\int_{-m}^{l}\varepsilon_0\Big(\big|\varphi'-|\xi|\psi\big|^2+\big|\psi'-|\xi|\varphi\big|^2+\frac{1}{3}\big|\psi'+|\xi|\varphi\big|^2\Big).
\end{aligned}\tag{3.143}$$

For $|\xi|\ge\xi_c$, the expression for Z is nonpositive, so the previous inequality holds obviously, and so we deduce that it holds for all $|\xi|>0$.

Together (3.141) with (3.143) for fixed ξ, we find

$$\frac{g[[\rho_0]]-\sigma|\xi|^2}{2}|\widehat{v}_3|^2-\int_{-m}^{l}\frac{p'(\rho_0)\rho_0}{2}\left|i\xi_1\widehat{v}_1+i\xi_2\widehat{v}_2+\partial_3\widehat{v}_3-\frac{g}{p'(\rho_0)}\widehat{v}_3\right|^2\mathrm{d}x_3$$
$$\leq\frac{\Lambda^2}{2}\int_{-m}^{l}\rho|\widehat{v}|^2+\frac{\Lambda}{2}\int_{-m}^{l}|i\xi_1\widehat{v}_1+i\xi_2\widehat{v}_2+\partial_3\widehat{v}_3|^2+\frac{\varepsilon_0}{2}|\widehat{B}|^2,\tag{3.144}$$

where $B=Dv+Dv^{\mathrm{T}}-2(\mathrm{div}v)I/3$. Integrating both sides of this inequality on R^2 and using (3.79), we prove the result.

When $\sigma>0$ and L is sufficiently small, a better result is available in the periodic case.

Lemma 3.5 *Let $v\in H^1(\Omega)$ be so that $v(x_1,x_2,-m)=v(x_1,x_2,l)=0$ and suppose in the periodic case that L satisfies* (3.58). *Then*

$$\int_{(2\pi(T))^2}\frac{g[[\rho_0]]}{2}|v_3|^2-\frac{\sigma}{2}|\nabla_{x_1x_2}v_3|^2-\int_{\Omega}\frac{p'(\rho_0)\rho_0}{2}\left|\mathrm{div}v-\frac{g}{p'(\rho_0)}v_3\right|^2\leq 0.\tag{3.145}$$

Proof Apply the horizontal Fourier transform to see that

$$4\pi^2L^2\int_{(2\pi(T))^2}\frac{g[[\rho_0]]}{2}|v_3|^2-\frac{\sigma}{2}|\nabla_{x_1x_2}v_3|^2-4\pi^2L^2\int_{\Omega}\frac{p'(\rho_0)\rho_0}{2}\left|\mathrm{div}v-\frac{g}{p'(\rho_0)}v_3\right|^2$$
$$=\sum_{\xi\in(L^{-1}Z)^2}\frac{g[[\rho_0]]-\sigma|\xi|^2}{2}|\widehat{v}_3|^2-\sum_{\xi\in(L^{-1}Z)^2}\int_{-m}^{l}\frac{p'(\rho_0)\rho_0}{2}\left|i\xi_1\widehat{v}_1+i\xi_2\widehat{v}_2+\partial_3\widehat{v}_3-\frac{g}{p'(\rho_0)}\widehat{v}_3\right|^2\mathrm{d}x_3.\tag{3.146}$$

Because of (3.58), there exists a unique $\xi\in(L^{-1}Z)^2$ such that $g[[\rho_0]]-g|\xi|^2\geq 0$, that is $\xi=0$. Obviously, if $\xi\neq 0$, there is

$$\frac{g[[\rho_0]]}{2}|\widehat{v}_3|^2-\int_{-m}^{l}\frac{p'(\rho_0)\rho_0}{2}\left|\partial_3\widehat{v}_3-\frac{g}{p'(\xi)}\widehat{v}_3\right|^2\mathrm{d}x_3\leq 0.\tag{3.147}$$

For this we expand the term in the integral and integrate by parts to get

$$\frac{g[[\rho_0]]}{2}|\widehat{v}_3|^2-\int_{-m}^{l}\frac{p'(\rho_0)\rho_0}{2}\left|\partial_3\widehat{v}_3-\frac{g}{p'(\xi)}\widehat{v}_3\right|^2\mathrm{d}x_3=-\frac{1}{2}\int_{-m}^{l}p'(\rho_0)\rho_0|\partial_3\widehat{v}_3|^2,\tag{3.148}$$

which yields the desired inequality.

3.7 Proof of Theorems 3.4 and 3.5

Proof of Theorem 3.4 Again, we will only prove the nonperiodic case. Integrate (3.136) with t from 0 to t to find that

$$\int_{\Omega}\rho_0\frac{|\partial_t v(t)|^2}{2}+\int_0^t\int_{\Omega}\frac{\varepsilon_0}{2}\left|D\partial_t v(s)+D\partial_t v(s)^{\mathrm{T}}-\frac{2}{3}(\mathrm{div}\partial_t v(s))\right|^2+\delta_0|\mathrm{div}\partial_t v(s)|\mathrm{d}s$$
$$\leq K_0+\int_{\mathbb{R}^2}\frac{g[[\rho_0]]}{2}|v_3(t)|^2-\frac{\sigma}{2}|\nabla_{x_1x_2}v_3(t)|^2-\int_{\Omega}\frac{p'(\rho_0)\rho_0}{2}\left|\mathrm{div}v(t)-\frac{g}{p'(\rho_0)}v_3(t)\right|^2,\tag{3.149}$$

where

$$K_0=\int_\Omega \rho_0\frac{|\partial_t v(0)|^2}{2}+\int_\Omega \frac{p'(\rho_0)\rho_0}{2}\Big|\mathrm{div}v(0)-\frac{g}{p'(\rho_0)}v_3(0)\Big|^2+\frac{\sigma}{2}|\nabla_{x_1x_2}v_3(0)|^2. \quad (3.150)$$

We may then apply Lemma 3.4 to get an inequality

$$\int_\Omega \rho_0\frac{|\partial_t v(t)|^2}{2}+\int_0^t\int_\Omega \frac{\varepsilon}{2}\Big(Dv_t(s)+D\partial_t v(s)^{\mathrm{T}}-\frac{2}{3}(\mathrm{div}\partial_t v(s))I\Big)+\delta_0|\mathrm{div}\partial_t v(s)|^2\mathrm{d}s$$
$$\le K_0+\frac{\Lambda^2}{2}\int_\Omega \rho_0|v(t)|^2+\frac{\Lambda}{2}\int_\Omega \frac{\varepsilon_0}{2}\Big(D\partial_t v(s)+D\partial_t v(s)^{\mathrm{T}}-\frac{2}{3}(\mathrm{div}\partial_t v(s))I\Big)+\delta_0|\mathrm{div}\partial_t v(t)|^2. \quad (3.151)$$

Using the definitions of the norms $\|\cdot\|_1$ and $\|\cdot\|_2$, we may compactly rewrite the previous inequality as

$$\frac{1}{2}\|\partial_t v(t)\|_1^2+\int_0^t\|\partial_t v(s)\|_2^2\mathrm{d}s\le K_0+\frac{\Lambda^2}{2}\|v(t)\|_1^2+\frac{\Lambda}{2}\|v(t)\|_2^2. \quad (3.152)$$

Integrati ng (3.152) with t and using Cauchy inequality, we obtain

$$\begin{aligned}\Lambda\|v(t)\|_2^2&=\Lambda\|v(0)\|_2^2+\Lambda\int_0^t 2(v(s),\partial_t v(s))\mathrm{d}s\\ &\le \Lambda\|v(0)\|_2^2+\int_0^t\|\partial_t v(s)\|_2^2\mathrm{d}s+\Lambda^2\int_0^t\|v(s)\|_1^2\mathrm{d}s. \end{aligned}\quad (3.153)$$

On the other hand, note that

$$\Lambda\partial_t\|v(s)\|_1^2=2\Lambda(\partial_t v(t),v(t))_1\le\Lambda^2\|v(s)\|_1^2+\|\partial_t v(s)\|_1^2. \quad (3.154)$$

From (3.152)-(3.154), we derive the following differential inequality

$$\partial_t\|v(t)\|_1^2+\|v(t)\|_2^2\le K_1+2\Lambda\|v(t)\|_1^2+2\Lambda\int_0^t\|v(s)\|_2^2\mathrm{d}s, \quad (3.155)$$

for $K_1=2K_0/\Lambda+2\|v(0)\|_2^2$. An application of Gronwall's theorem then shows that

$$\|v(t)\|_1^2+\int_0^t\|v(s)\|_2^2\mathrm{d}s\le \mathrm{e}^{2\Lambda t}\|v(0)\|_1^2+\frac{K_1}{2\Lambda}(\mathrm{e}^{2\Lambda t}-1). \quad (3.156)$$

To derive the corresponding bounds for $\|v(t)\|_2^2,\|\partial_t v(t)\|_1^2$, according (3.152), (3.153) and (3.156), we obtain

$$\frac{1}{\Lambda}\|\partial_t v(t)\|_1^2+\|v(t)\|_2^2\le K_1+\Lambda\|v(t)\|_1^2+2\Lambda\int_0^t\|v(s)\|_2^2\mathrm{d}s\le\mathrm{e}^{2\Lambda t}(2\Lambda\|v(0)\|_1^2+K_1), \quad (3.157)$$

$$K_0\le C(\|\partial_t v(0)\|_1^2+\|v(0)\|_1^2+\|v(0)\|_2^2)+\sigma\int_{\mathbb{R}^2}|\nabla_{x_1x_2}v_3(0)|^2, \quad (3.158)$$

for a constant $C>0$ depending on $\rho_0^{\pm},p_{\pm},\Lambda,\varepsilon_{\pm},\delta_{\pm},\sigma,g,m,l$.

Proof of Theorem 3.5 We again integrate (3.136) with t from 0 to t to find that

$$\int_\Omega \rho_0 \frac{|\partial_t v(t)|^2}{2} + \int_0^t \int_\Omega \frac{\varepsilon_0}{2}\Big|Dv_t(s) + D\partial_t v(s)^{\mathrm{T}} - \frac{2}{3}\mathrm{div}\partial_t v(s) I\Big|^2 + \delta_0 |\mathrm{div}\partial_t v(s)|^2 \mathrm{d}s$$
$$\leq K_1 + \int_{(2\pi LT)^2} \frac{g[[\rho_0]]}{2}|v_3(t)|^2 - \frac{\sigma}{2}|\nabla_{x_1x_2} v_3(t)|^2 - \int_\Omega \frac{p'(\rho_0)\rho_0}{2}\Big|\mathrm{div} v(t) - \frac{g}{p'(\rho_0)} v_3(t)\Big|^2.$$

We may apply Lemma 3.5 to see that the sum of all of the integrals on the right side of the previous inequality are nonpositive, and hence

$$\frac{1}{2}\|\partial_t v(t)\|_1^2 + \int_0^t \|\partial_t v(s)\|_2^2 \mathrm{d}s \leq K_1. \tag{3.159}$$

From this we deduce that

$$\|v(t)\|_1 + \|v(t)\|_2 \leq \|v(0)\|_1 + \|v(0)\|_2 + 3\sqrt{t}\sqrt{K_1}. \tag{3.160}$$

Then, using $\partial_t \eta = v$, we get

$$\|\eta(t)\|_1 + \|\eta(t)\|_2 \leq \|\eta(0)\|_1 + \|\eta(0)\|_2 + t(\|v(0)\|_1 + \|v(0)\|_2) + 2t^{\frac{3}{2}}\sqrt{K_1}. \tag{3.161}$$

To derive the estimates for $\partial_t^j v$ $(j \geq 2)$ we apply ∂^j to (3.133). Then $w = \partial^j v$ satisfies the same equation and boundary conditions as v. Similarly we derive the inequality

$$\frac{1}{2}\|\partial_t^j v(t)\|_1^2 + \int_0^t \|\partial_t^j v(s)\|_2^2 \mathrm{d}s \leq K_3, \quad \text{for any } j \geq 1.$$

This implies (3.77) holds. Note that

$$\begin{aligned}
\|\partial_t^j v(t)\|_2^2 &\leq \|\partial_t^j v(0)\|_2^2 + 2\int_0^t \|\partial_t^j v(s)\|_2 \|\partial_t^{j+1} v(s)\|_2 \mathrm{d}s \\
&\leq \|\partial_t^j v(0)\|_2^2 + 2\left(\int_0^t \|\partial_t^j v(s)\|_2^2 \mathrm{d}s\right)^{\frac{1}{2}} \left(\int_0^t \|\partial_t^{j+1} v(s)\|_2^2\right)^{\frac{1}{2}} \\
&\leq \|\partial_t^j v(0)\|_2^2 + 2\sqrt{K_j}\sqrt{K_{j+1}}.
\end{aligned} \tag{3.162}$$

Thus (3.78) holds.

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