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COUPLED NONLOCAL NONLINEAR SCHRÖDINGER EQUATION AND N-SOLITON SOLUTION FORMULA WITH DARBOUX TRANSFORMATION*†

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Abstract

Ablowitz and Musslimani proposed some new nonlocal nonlinear integrable equations including the nonlocal integrable nonlinear Schrödinger equation. In this paper, we investigate the Darboux transformation of coupled nonlocal nonlinear Schrödinger (CNNLS) equation with a spectral problem. Starting from a special Lax pairs, the CNNLS equation is constructed. Then, we obtain the one-, two- and N-soliton solution formulas of the CNNLS equation with N-fold Darboux transformation. Based on the obtained solutions, the propagation and interaction structures of these multi-solitons are shown, the evolution structures of the one-dark and one-bright solitons are exhibited with $N = 1$, and the overtaking elastic interactions among the two-dark and two-bright solitons are considered with $N = 2$. The obtained results are different from those of the solutions of the local nonlinear equations. Some different propagation phenomena can also be produced through manipulating multi-soliton waves. The results in this paper might be helpful for understanding some physical phenomena described in plasmas.

Keywords coupled nonlocal nonlinear Schrödinger (CNNLS) equation; Darboux transformation; dark soliton; bright soliton

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1 Introduction

The Schrödinger equation is one of the basic equations of quantum mechanics proposed by physicist Schrödinger in 1926. In recently, Ablowitz and Musslimani [1,2] proposed some new nonlocal nonlinear integrable equations which include the

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nonlocal integrable nonlinear Schrödinger equation, mKdV equation, and so on. According to the relative scale of the relative length of the root beam width and the nonlinear response function of the medium, the nonlocal nonlinear Schrödinger equation can be divided into four classes [3] including the local class, weakly nonlocal class, general nonlocal class and strongly nonlocal class. The spatial solitons in nonlocal nonlinear media have attracted great interesting [4-13]. The research status of nonlocal spatial solitons were summarized and reviewed in [11].

With the further study of the soliton theory, it provides many methods for solving nonlinear partial differential equations, such as the homogeneous balance method [14], bilinear method [15], traveling wave method [15], Darboux transformation (DT) method [16], inverse scattering transform method [15-19]. For examples, some discrete rogue-wave solutions with dispersion in parity-time symmetric potential of Ablowitz-Musslimani equation were derived in [20]. Some bright, dark and breather wave soliton solutions of the super-integrable hierarchy were presented by Darboux transformation [21]. The non-autonomous multi-rogue wave solutions in a spin-1 coupled nonlinear Gross-Pitaevskii equation with varying dispersions, higher nonlinearities, gain/loss and external potentials were investigated in [22]. The generalized three-coupled Gross-Pitaevskii equations by means of the DT and Hirota's method were worked, and several non-autonomous matter-wave solitons including dark-dark-dark and bright-bright-bright shapes were obtained in [23]. The non-autonomous discrete vector bright-dark solutions and their controllable behaviors in the coupled Ablowitz-Ladik equation with variable coefficients were considered in [24]. The Darboux transformation method with 4×4 spectral problem are applied to study a specific equation and then the explicit solutions of the lattice integrable coupling equation were obtained in [25, 26].

The spectral problem stems from a solution of nonlinear partial differential equations, and the new solution was derived by Darboux transformation method [16]. The Darboux transformation can get a new solution from a known equation, also some multi-soliton solutions of the nonlinear partial differential equation can be obtained through multiple Darboux transformation [27-32]. The coupled nonlinear schrödinger equation, which describes a nonlinear diffusion regularity of two nonlinear wave propagation in the medium, not only is applied widely in the field of nonlinear optics, but also plays an important role in meteorology.

Wu and He generated the derivative nonlinear Schrödinger (NLS) equations, whose nonlocal extensions are from Lie algebra splittings and automorphisms in [33]. A chain of nonsingular localized-wave solutions was derived for a nonlocal NLS equation with the self-induced parity-time (PT)-symmetric potential through the N-th Darboux transformation by Li and Xu in [34]. Some rational soliton solutions were derived for the parity-time-symmetric nonlocal NLS model with the defocusing-type nonlinearity by the generalized Darboux transformation in [35], which includes the first-order solution, dark-antidark, and antidark-dark soliton. Zhang, Qiu, Cheng and He derived a kind of rational solution with two free phase parameters of nonlocal NLS equation, which satisfies the parity-time (PT) symmetry condition in [36]. Some new unified two-parameter wave model, connecting integrable local and nonlocal vector NLS equations were presented by Yan in [37]. Ma and Zhu introduced the geometry of a nonlocal NLS equation and its discrete version in [38].

In this paper, we investigate the Darboux transformation of a coupled nonlocal nonlinear Schrödinger (CNNLS) equation. Starting from a special Lax pairs, we construct a nontrivial single soliton solution from the zero solution, and a double soliton solution from a single soliton solution, finally obtain the N-soliton solution formula of the CNNLS equation with N-fold Darboux transformation. In addition, we find that the coupled nonlocal soliton equations have some richer mathematical structures, the propagation and interaction structures than a single equation.

This paper is organized as follows. In Section 2, we construct Darboux transformation for CNNLS equation, and prove the procedure of DT. In Section 3, we apply the Darboux transformation to CNNLS and obtain N-soliton solution formula. The evolutions of the intensity distribution of the soliton solutions are illustrated in form of figures.

2 Darboux transformation for the coupled nonlocal Schrödinger equation

There are some important results of coupled Schrödinger (CNLS) equation in the previous works. In this section, we are motivated by the investigation for the general coupled CNLS equation and consider the coupled nonlocal Schrödinger (CNNLS) equation as follows:

$$\begin{cases} ip_t+p_{xx}+2[app^*(-x,t)+cqq^*(-x,t)+bpq^*(-x,t)+b^*qp^*(-x,t)]p=0, \\ iq_t+q_{xx}+2[app^*(-x,t)+cqq^*(-x,t)+bpq^*(-x,t)+b^*qp^*(-x,t)]q=0. \end{cases} \tag{1}$$

We first demonstrate the integrability of the CNNLS equation (1), then we construct its Darboux transformation. The Lax pairs of CNNLS equation (1) is given as follows

$$\varphi_x = U\varphi = (i\lambda\sigma_3 + p)\varphi = \begin{pmatrix} i\lambda & p & q \\ -r_1 & -i\lambda & 0 \\ -r_2 & 0 & -i\lambda \end{pmatrix} \varphi, \tag{2}$$

and

$$\begin{aligned}\varphi_t = V\varphi &= -(2i\lambda^2 + 2\lambda p + i(p^2+p_x)\sigma_3)\varphi \\ &= \begin{pmatrix} -2i\lambda^2 + ir_1p + ir_2q & -2\lambda p + ip_x & -2\lambda q + iq_x \\ 2\lambda r_1 + ir_{1,x} & 2i\lambda^2 - ir_1p & -ir_1q \\ 2\lambda r_2 + ir_{2,x} & -ir_2p & 2i\lambda^2 - ir_2q \end{pmatrix}\varphi, \end{aligned} \tag{3}$$

where

$$P = \begin{pmatrix} 0 & p & q \\ -r_1 & 0 & 0 \\ -r_2 & 0 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{4}$$

and $r_1 = ap^*(-x,t) + bq^*(-x,t)$, $r_2 = b^*p^*(-x,t) + cq^*(-x,t)$, $\sigma_3 P = -P\sigma_3$, and $\sigma_3 P^2 \sigma_3 = P^2$. $p(x,t)$, $q(x,t)$, $r_1(x,t)$, $r_2(x,t)$ are the "potential" about x and t, a and c are real, and b is complex. We can check that the zero-curvature equation $U_t - V_x + [U,V] = 0$ leads to equation (1) under the condition of compatibility.

Next, we construct the Darboux transformation of CNNLS equation with the Lax pairs of equations (2) and (3), which are satisfied with the 3×3 transformation matrix of φ, $\widetilde{U}$ and $\widetilde{V}$. We consider the isospectral problem of CNNLS equation and recommend a gauge transformation T of the Lax pairs of equations (2) and (3):

$$\widetilde{\varphi_n} = T\varphi_n, \quad T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}, \tag{5}$$

$$\varphi_x = \widetilde{U}\varphi, \quad \widetilde{U} = (T_x + TU)T^{-1}, \tag{6}$$

$$\varphi_t = \widetilde{V}\varphi, \quad \widetilde{V} = (T_t + TV)T^{-1}. \tag{7}$$

If $\widetilde{U}$, $\widetilde{V}$, U and V have the same types, then system (5) is called Darboux transformation of the CNNLS equation. $\psi = (\psi_1,\psi_2,\psi_3)^{\mathrm{T}}$, $\phi = (\phi_1,\phi_2,\phi_3)^{\mathrm{T}}$, $X = (X_1,X_2,X_3)^{\mathrm{T}}$ are three basic solutions of systems (2) and (3), then we give the following linear algebraic systems:

$$\begin{cases} \displaystyle\sum_{i=0}^{N-1}(A_{11}^{(i)} + A_{12}^{(i)}M_j^{(1)} + A_{13}^{(i)}M_j^{(2)})\lambda_j^i = -\lambda_j^N, \\ \displaystyle\sum_{i=0}^{N-1}(A_{21}^{(i)} + A_{22}^{(i)}M_j^{(1)} + A_{23}^{(i)}M_j^{(2)})\lambda_j^i = -M_j^{(1)}\lambda_j^N, \\ \displaystyle\sum_{i=0}^{N-1}(A_{31}^{(i)} + A_{32}^{(i)}M_j^{(1)} + A_{33}^{(i)}M_j^{(2)})\lambda_j^i = -M_j^{(2)}\lambda_j^N, \end{cases} \tag{8}$$

with

$$M_j^{(1)} = \frac{\psi_2 + \nu_j^{(1)}\phi_2 + \nu_j^{(2)}X_2}{\psi_1 + \nu_j^{(1)}\phi_1 + \nu_j^{(2)}X_1}, \quad M_j^{(2)} = \frac{\psi_3 + \nu_j^{(1)}\phi_3 + \nu_j^{(2)}X_3}{\psi_1 + \nu_j^{(1)}\phi_1 + \nu_j^{(2)}X_1}, \quad 0 \le j \le 3N, \tag{9}$$

where λ_j and $\nu_j^{(k)}$ $(i \neq k,\ \lambda_i \neq \lambda_j,\ \nu_i^{(k)} \neq \nu_j^{(k)},\ k=1,2)$ should choose appropriate parameters, here the determinants of coefficients for equation (8) are nonzero.

We give a 3×3 matrix T, which is of the form as follows

$$\begin{cases} T_{11} = \lambda^N + \sum\limits_{i=0}^{N-1} A_{11}^{(i)} \lambda^i, \quad T_{12} = \sum\limits_{i=0}^{N-1} A_{12}^{(i)} \lambda^i, \quad T_{13} = \sum\limits_{i=0}^{N-1} A_{13}^{(i)} \lambda^i, \\ T_{21} = \sum\limits_{i=0}^{N-1} A_{21}^{(i)} \lambda^i, \quad T_{22} = \lambda^N + \sum\limits_{i=0}^{N-1} A_{22}^{(i)} \lambda^i, \quad T_{23} = \sum\limits_{i=0}^{N-1} A_{23}^{(i)} \lambda^i, \\ T_{31} = \sum\limits_{i=0}^{N-1} A_{31}^{(i)} \lambda^i, \quad T_{32} = \sum\limits_{i=0}^{N-1} A_{32}^{(i)} \lambda^i, \quad T_{33} = \lambda^N + \sum\limits_{i=0}^{N-1} A_{33}^{(i)} \lambda^i, \end{cases} \tag{10}$$

where N is a natural number, A_{mn}^i $(m,n = 1,2,3,\ m \geq 0)$ are the functions of x and t. By calculations, we can obtain ΔT as follows

$$\Delta T = \prod_{j=1}^{3N} (\lambda - \lambda_j), \tag{11}$$

which proves that λ_j $(1 \leq j \leq 3N,)$ are $3N$ roots of ΔT. Based on the above conditions, we will prove that $\widetilde{U}$ and $\widetilde{V}$ have the same forms with U and V, respectively.

Proposition 1 *The matrix* $\widetilde{U}$ *defined by* (6) *has the same type as* U*, then we have*

$$\widetilde{U} = \begin{pmatrix} i\lambda & \widetilde{p} & \widetilde{q} \\ -\widetilde{r}_1 & -i\lambda & 0 \\ -\widetilde{r}_2 & 0 & -i\lambda \end{pmatrix}, \tag{12}$$

in which the transformation formulas between old and new potentials are shown on as follows

$$\begin{cases} \widetilde{p} = p - 2iA_{12}, \\ \widetilde{q} = q - 2iA_{13}. \end{cases} \tag{13}$$

The transformations (13) *are used to get a Darboux transformation of the spectral problem* (6).

Proof Setting $T^{-1} = \frac{T^*}{\Delta T}$ and

$$(T_x + TU)T^* = \begin{pmatrix} B_{11}(\lambda) & B_{12}(\lambda) & B_{13}(\lambda) \\ B_{21}(\lambda) & B_{22}(\lambda) & B_{23}(\lambda) \\ B_{31}(\lambda) & B_{32}(\lambda) & B_{33}(\lambda) \end{pmatrix}, \tag{14}$$

we are easy to verify that B_{sl} $(1 \leq s,l \leq 3)$ are $3N$-order or $3N+1$-order polynomials in λ.

By some simply calculations, λ_j $(1 \le j \le 3)$ are the roots of B_{sl} $(1 \le s, l \le 3)$. Hence, equation (14) has the following structure

$$(T_x + TU)T^* = (\Delta T)C(\lambda), \tag{15}$$

where

$$C(\lambda) = \begin{pmatrix} C_{11}^{(1)}\lambda + C_{11}^{(0)} & C_{12}^{(0)} & C_{13}^{(0)} \\ C_{21}^{(0)} & C_{22}^{(1)}\lambda + C_{22}^{(0)} & C_{23}^{(0)} \\ C_{31}^{(0)} & C_{32}^{(0)} & C_{33}^{(1)}\lambda + C_{33}^{(0)} \end{pmatrix}, \tag{16}$$

and $C_{mn}^{(k)}$ $(m, n = 1, 2,\ k = 0, 1)$ satisfy the functions independent of λ. Equation (15) is obtained as follows

$$(T_x + TU) = C(\lambda)T. \tag{17}$$

Through comparing the coefficients of λ in equation (17), we have the following system

$$\begin{cases} C_{11}^{(1)} = i, \quad C_{11}^{(0)} = 0, \quad C_{12}^{(0)} = p - 2iA_{12} = \widetilde{p}, \quad C_{13}^{(0)} = q - 2iA_{13} = \widetilde{q}, \\ C_{21}^{(0)} = 2iA_{21} - r_1 = -\widetilde{r}_1, \quad C_{22}^{(1)} = -i, \quad C_{22}^{(0)} = 0, \quad C_{23}^{(0)} = 0, \\ C_{31}^{(0)} = -r_2 + 2iA_{31} = -\widetilde{r}_2, \quad C_{32}^{(0)} = 0, \quad C_{33}^{(1)} = -i, \quad C_{33}^{(0)} = 0. \end{cases} \tag{18}$$

In the following section, we assume that the new matrix $\widetilde{U}$ has the same type with U, which means that they have the same structures only by transform p, q, r_1, r_2 of U into $\widetilde{p}, \widetilde{q}, \widehat{r}_1, \widetilde{r}_2$ of $\widetilde{U}$. After detailed calculation, we compare the ranks of λ^N, and get the objective equations as follows:

$$\begin{cases} \widetilde{p} = p - 2iA_{12}, \\ \widetilde{q} = q - 2iA_{13}. \end{cases} \tag{19}$$

From equations (12) and (13), we know that $\widetilde{U} = C(\lambda)$. The proof is completed.

Proposition 2 *Under the transformation* (19)*, the matrix* $\widetilde{V}$ *defined by* (7) *has the same form as* V*, that is,*

$$\widetilde{V} = \begin{pmatrix} -2i\lambda^2 + i\widetilde{r}_1\widetilde{p} + i\widetilde{r}_2\widetilde{q} & -2\lambda\widetilde{p} + i\widetilde{p}_x & -2\lambda\widetilde{q} + i\widetilde{q}_x \\ 2\lambda\widetilde{r}_1 + i\widetilde{r}_{1,x} & 2i\lambda^2 - i\widetilde{r}_1\widetilde{p} & -i\widetilde{r}_1\widetilde{q} \\ 2\lambda\widetilde{r}_2 + i\widetilde{r}_{2,x} & -i\widetilde{r}_2 p & 2i\lambda^2 - i\widetilde{r}_2\widetilde{q} \end{pmatrix}. \tag{20}$$

Proof We assume the new matrix $\widetilde{V}$ also have the same form with V. If we obtain some relations between p, q, r_1, r_2 and $\widetilde{p}, \widetilde{q}, \widehat{r}_1, \widetilde{r}_2$ similar to (13), we can prove that the gauge transformation T turns the Lax pairs U, V into new Lax pairs $\widetilde{U}, \widetilde{V}$ with the same types.

Set $T^{-1} = \frac{T^*}{\Delta T}$ and

$$(T_t+TV)T^*=\begin{pmatrix}E_{11}(\lambda) & E_{12}(\lambda) & E_{13}(\lambda)\\ E_{21}(\lambda) & E_{22}(\lambda) & E_{23}(\lambda)\\ E_{31}(\lambda) & E_{32}(\lambda) & E_{33}(\lambda)\end{pmatrix}. \tag{21}$$

It is easy to verify that E_{sl} $(1\le s,l\le 3)$ are $3N$+1-order or $3N$+2-order polynomials in λ.

Through some calculations, λ_j $(1\le j\le 3)$ are the roots of E_{sl} $(1\le s,l\le 3)$. Thus, equation (21) has the following structure

$$(T_t+TV)T^*=(\Delta T)F(\lambda), \tag{22}$$

where

$$F(\lambda)=\begin{pmatrix}F_{11}^{(2)}\lambda^2+F_{11}^{(1)}\lambda+F_{11}^{(0)} & F_{12}^{(1)}\lambda+F_{12}^{(0)} & F_{13}^{(1)}\lambda+F_{13}^{(0)}\\ F_{21}^{(1)}\lambda+F_{21}^{(0)} & F_{22}^{(2)}\lambda^2+F_{22}^{(1)}\lambda+F_{22}^{(0)} & F_{23}^{(1)}\lambda+F_{23}^{(0)}\\ F_{31}^{(1)}\lambda+F_{31}^{(0)} & F_{32}^{(1)}\lambda+F_{32}^{(0)} & F_{33}^{(2)}\lambda^2+F_{33}^{(1)}\lambda+F_{33}^{(0)}\end{pmatrix}, \tag{23}$$

and $F_{mn}^{(k)}$ $(m,n=1,2,3,\ k=0,1,2)$ satisfy the functions without λ. According to equation (23), we obtain the following equation

$$T_t+TV=F(\lambda)T. \tag{24}$$

Through comparing the coefficients of λ in equation (24), we get the objective equations as follows:

$$\begin{cases}F_{11}^{(2)}=-2i,\quad F_{11}^{(1)}=0,\\ F_{11}^{(0)}=ipr_1+iqr_2+2r_1A_{12}+2r_2A_{13}-2\widetilde{p}A_{21}-2\widetilde{q}A_{31}=i\widetilde{p}\widetilde{r}_1+i\widetilde{q}\widetilde{r}_2,\\ F_{12}^{(1)}=p-2iA_{12}=\widetilde{p},\quad F_{12}^{(0)}=i(p_x+2ipA_{11}-2i\widetilde{q}-2i\widetilde{p}A_{22})=i\widetilde{p}_x,\\ F_{13}^{(1)}=q-2iA_{13}=\widetilde{q},\quad F_{13}^{(0)}=i(q_x+2iqA_{11}-2i\widetilde{p}-2i\widetilde{q})=i\widetilde{q}_x,\\ F_{21}^{(1)}=r_1-2iA_{21}=\widetilde{r}_1,\quad F_{21}^{(0)}=i(r_{1,x}-2ir_1A_{22}+2iA_{11}\widetilde{r}_1-2iA_{23}\widetilde{r}_2)=i\widetilde{r}_{1,x},\\ F_{22}^{(2)}=2i,\quad F_{22}^{(1)}=0,\\ F_{22}^{(0)}=-2pA_{21}-ir_1p-2A_{12}\widetilde{r}_1=-i\widetilde{r}_1\widetilde{p},\\ F_{23}^{(1)}=0,\quad F_{23}^{(0)}=-2qA_{21}-ir_1q-2\widetilde{r}_1A_{13}=-\widetilde{r}_1\widetilde{q},\\ F_{31}^{(1)}=r_2-2iA_{31}=\widetilde{r}_2,\quad F_{31}^{(0)}=ir_{2,x}-2ir_1A_{32}-2ir_2A_{33}-2i\widetilde{r}_2A_{11}=i\widetilde{r}_{2,x},\\ F_{32}^{(1)}=0,\quad F_{32}^{(0)}=-i(r_2-2iA_{31})(p-2iA_{12})=-i\widetilde{r}_2\widetilde{p},\\ F_{33}^{(2)}=2i,\quad F_{33}^{(1)}=0,\quad F_{33}^{(0)}=-i(r_2-2iA_{31})(p-2iA_{12})=-i\widetilde{r}_2\widetilde{q}.\end{cases} \tag{25}$$

In the above section, we assume the new matrix $\widetilde{V}$ has the same type with V, which means they have the same structures only by transform p,q,r_1,r_2 of V into $\widetilde{p},\widetilde{q},\widehat{r}_1,\widetilde{r}_2$ of $\widetilde{V}$. From equations (13) and (19), we know that $\widetilde{V}=F(\lambda)$. The proof is completed.

3 N-soliton Solution Formula for the Coupled Nonlocal Schrödinger Equation

In order to obtain the N-soliton solution formula of CNNLS equation with Darboux transformation, we firstly give a set of seed solutions $p = q = r_1 = r_2 = 0$ and substitute the solutions into equations (2) and (3), which can get three basic solutions of CCNLS equation:

$$\psi_{(\lambda)} = \begin{pmatrix} e^{i\lambda x - 2i\lambda^2 t} \\ 0 \\ 0 \end{pmatrix}, \quad \phi_{(\lambda)} = \begin{pmatrix} 0 \\ e^{-i\lambda x + 2i\lambda^2 t} \\ 0 \end{pmatrix}, \quad X_{(\lambda)} = \begin{pmatrix} 0 \\ 0 \\ e^{-i\lambda x + 2i\lambda^2 t} \end{pmatrix}. \tag{26}$$

Substituting equation (26) into equation (9), we obtain

$$\begin{cases} M_j^{(1)} = \dfrac{\nu_j^{(1)} e^{-i\lambda x + 2i\lambda^2 t}}{e^{i\lambda x - 2i\lambda^2 t}} = \nu_j^{(1)} e^{-2i\lambda x + 4i\lambda^2 t}, \\ M_j^{(2)} = \dfrac{\nu_j^{(2)} e^{-i\lambda x + 2i\lambda^2 t}}{e^{i\lambda x - 2i\lambda^2 t}} = \nu_j^{(2)} e^{-2i\lambda x + 4i\lambda^2 t} \end{cases} \tag{27}$$

with $\nu_j^{(i)} = e^{2iF_j^{(i)}}$ $(1 \le i \le 2,\ 1 \le j \le 3N)$.

In order to obtain the one-soliton solution of equation (1). We consider $N = 1$ in equations (10) and (11), and obtain the matrix T

$$T = \begin{pmatrix} \lambda + A_{11} & A_{12} & A_{13} \\ A_{21} & \lambda + A_{22} & A_{23} \\ A_{31} & A_{32} & \lambda + A_{33} \end{pmatrix}, \tag{28}$$

and

$$\begin{cases} \lambda_j + A_{11} + M_j^{(1)} A_{12} + M_j^{(2)} A_{13} = 0, \\ A_{21} + M_j^{(1)} (\lambda_j + A_{22}) + M_j^{(2)} A_{23} = 0, \\ A_{31} + M_j^{(1)} A_{32} + M_j^{(2)} (\lambda_j + A_{33}) = 0. \end{cases} \tag{29}$$

Let $N_k^{(1)} = e^{-2i\lambda_k x + 4i\lambda_k^2 t + 2iF_k^{(1)}}$ and $N_k^{(2)} = e^{-2i\lambda_k x + 4i\lambda_k^2 t + 2iF_k^{(2)}}$. According to equation (8), we can obtain

$$\begin{gathered} \Delta = \begin{vmatrix} 1 & N_1^{(1)} & N_1^{(2)} \\ 1 & N_2^{(1)} & N_2^{(2)} \\ 1 & N_3^{(1)} & N_3^{(2)} \end{vmatrix}, \quad \Delta_{13} = \begin{vmatrix} 1 & N_1^{(1)} & -\lambda_1 \\ 1 & N_2^{(1)} & -\lambda_2 \\ 1 & N_3^{(1)} & -\lambda_3 \end{vmatrix}, \quad \Delta_{12} = \begin{vmatrix} 1 & -\lambda_1 & N_1^{(2)} \\ 1 & -\lambda_2 & N_2^{(2)} \\ 1 & -\lambda_3 & N_3^{(2)} \end{vmatrix}, \\ \Delta_{21} = \begin{vmatrix} -\lambda_1 N_1^{(1)} & N_1^{(1)} & N_1^{(2)} \\ -\lambda_2 N_2^{(1)} & N_2^{(1)} & N_2^{(2)} \\ -\lambda_3 N_3^{(1)} & N_3^{(1)} & N_3^{(2)} \end{vmatrix}, \quad \Delta_{31} = \begin{vmatrix} -\lambda_1 N_1^{(2)} & N_1^{(1)} & N_1^{(2)} \\ -\lambda_2 N_2^{(2)} & N_2^{(1)} & N_2^{(2)} \\ -\lambda_3 N_3^{(2)} & N_3^{(1)} & N_3^{(2)} \end{vmatrix}. \end{gathered} \tag{30}$$

Based on equations (9) and (27), we can obtain the following systems

$$A_{12} = \frac{\Delta_{12}}{\Delta}, \quad A_{13} = \frac{\Delta_{13}}{\Delta}, \quad A_{21} = \frac{\Delta_{21}}{\Delta}, \quad A_{31} = \frac{\Delta_{31}}{\Delta}, \tag{31}$$

where the analytic one-soliton solutions of CNNLS equation are obtained by the Darboux transformation method as follows

$$\widetilde{p} = -2i\frac{\Delta_{12}}{\Delta}, \quad \widetilde{q} = -2i\frac{\Delta_{13}}{\Delta}. \tag{32}$$

To illustrate the wave propagations of the obtained one-soliton solutions (32), we can choose these free parameters in the forms $\lambda_1, \lambda_2, \lambda_3, F_m^{(k)}$ $(m = 1, 2, 3,\ k = 1, 2, 3)$ and the intensity distributions for the soliton solutions given by equation (32) are illustrated in Figure 1. From the single soliton, we can find that the amplitude of the bright-dark soliton grows and decays with time depending on the parameters $\lambda_1, \lambda_2, \lambda_3,\ F_m^{(k)}$ $(m = 1, 2, 3,\ k = 1, 2, 3)$.

In equation (32), when $\lambda_1 = \lambda_2 = \lambda_3$ or $F_1^{(1)} = F_1^{(2)} = F_2^{(1)} = F_2^{(2)} = F_3^{(1)} = F_3^{(2)}$ in the first case, the denominator equation (30) is zero, also Δ is zero, so there is no solution. In the second case, when the values $\lambda_1, \lambda_2, \lambda_3$ are not exactly same and the values $F_1^{(1)}, F_1^{(2)}, F_2^{(1)}, F_2^{(2)}, F_3^{(1)}, F_3^{(2)}$ are not exactly same, then the equation has some solutions.

Now we consider $N = 2$ in equations (10) and (11), and obtain the following system

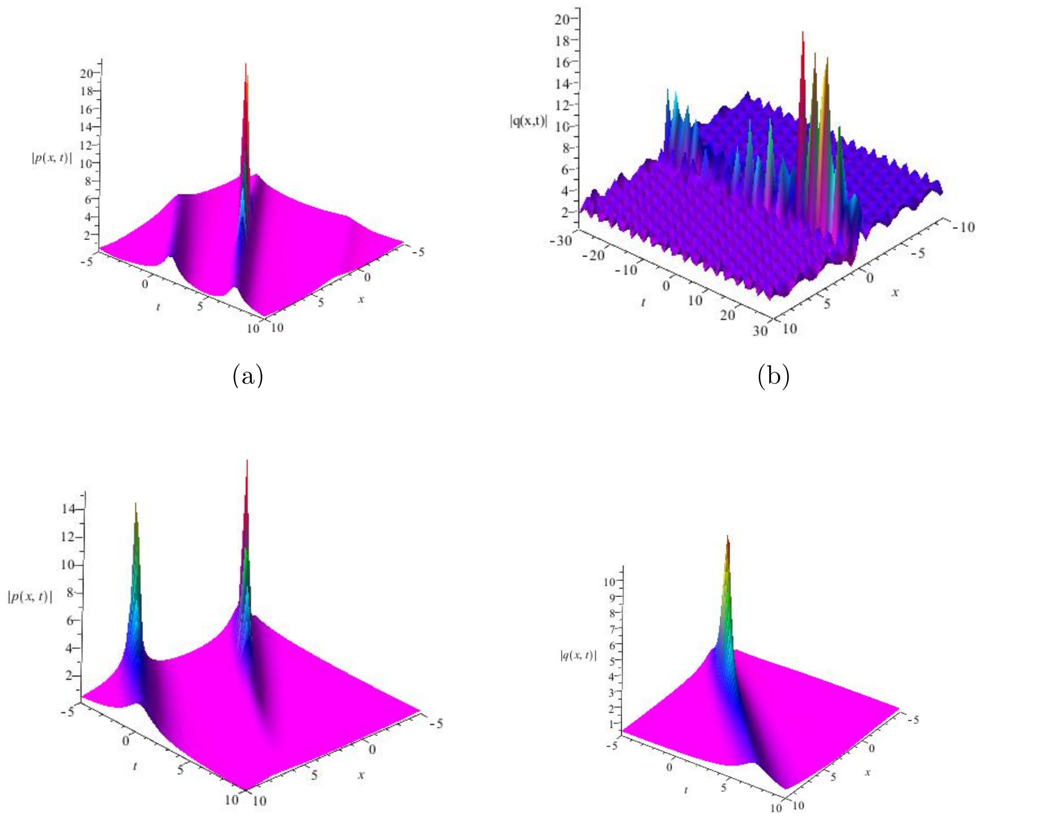
$$\begin{cases} \lambda_j^2 + A_{11}^{(0)} + M_j^{(1)}A_{12}^{(0)} + M_j^{(2)}A_{13}^{(0)} + (A_{11}^{(1)} + M_j^{(1)}A_{12}^{(1)} + M_j^{(2)}A_{13}^{(1)})\lambda_j = 0, \\ -\lambda_j^2 M_j^{(1)} + A_{21}^{(0)} + M_j^{(1)}A_{22}^{(0)} + M_j^{(2)}A_{23}^{(0)} + (A_{21}^{(1)} + M_j^{(1)}A_{22}^{(1)} + M_j^{(2)}A_{23}^{(1)})\lambda_j = 0, \\ -\lambda_j^2 M_j^{(2)} + A_{31}^{(0)} + M_j^{(1)}A_{32}^{(0)} + M_j^{(2)}A_{33}^{(0)} + (A_{31}^{(1)} + M_j^{(1)}A_{32}^{(1)} + M_j^{(2)}A_{33}^{(1)})\lambda_j = 0, \end{cases} \tag{33}$$

where $i = 0, 1, 2$ and $j = 1, 2, \cdots, 6$. According to equation (33) and Cramer's rule, we can obtain

$$A_{12} = \frac{\Delta_{12}}{\Delta}, \quad A_{13} = \frac{\Delta_{13}}{\Delta}, \quad A_{21} = \frac{\Delta_{21}}{\Delta}, \quad A_{31} = \frac{\Delta_{31}}{\Delta}, \tag{34}$$

where

$$\Delta = \begin{vmatrix} 1 & N_1^{(1)} & N_1^{(2)} & \lambda_1 & \lambda_1 N_1^{(1)} & \lambda_1 N_1^{(2)} \\ 1 & N_2^{(1)} & N_2^{(2)} & \lambda_2 & \lambda_2 N_2^{(1)} & \lambda_2 N_2^{(2)} \\ 1 & N_3^{(1)} & N_3^{(2)} & \lambda_3 & \lambda_3 N_3^{(1)} & \lambda_3 N_3^{(2)} \\ 1 & N_4^{(1)} & N_4^{(2)} & \lambda_4 & \lambda_4 N_4^{(1)} & \lambda_4 N_4^{(2)} \\ 1 & N_5^{(1)} & N_5^{(2)} & \lambda_5 & \lambda_5 N_5^{(1)} & \lambda_5 N_5^{(2)} \\ 1 & N_6^{(1)} & N_6^{(2)} & \lambda_6 & \lambda_6 N_6^{(1)} & \lambda_6 N_6^{(2)} \end{vmatrix},$$



(a) (b)

(c) (d)

Figure 1: (a) The intensity distribution $|\widetilde{p}|$ of equation (32) with $\lambda_1 = 0.2$, $\lambda_2 = 0.3$, $\lambda_3 = 0.5$, $F_1^{(1)} = 0.1 + 0.2i$, $F_2^{(2)} = 0.6i$, $F_1^{(2)} = 0.5 + 0.3i$, $F_2^{(1)} = 0.3 + 0.1i$, $F_3^{(1)} = 0.2 + 0.3i$, $F_3^{(2)} = 0.2 + 0.1i$; (b) the intensity distribution $|\widetilde{q}|$ of equation (32) with $\lambda_1 = 1$, $\lambda_2 = i$, $\lambda_3 = 2$, $F_1^{(1)} = i$, $F_2^{(2)} = 3$, $F_1^{(2)} = 2$, $F_2^{(1)} = 1$, $F_3^{(1)} = 2$, $F_3^{(2)} = 1$; (c) the intensity distribution $|\widetilde{p}|$ of equation (32) with $\lambda_1 = 0.2$, $\lambda_2 = 0.3$, $\lambda_3 = 0.1$, $F_1^{(1)} = 1.1 + 0.2i$, $F_2^{(2)} = 1.6i$, $F_1^{(2)} = 0.5 + 1.3i$, $F_2^{(1)} = 1.3 + 0.1i$, $F_3^{(1)} = 1.2 + 0.3i$, $F_3^{(2)} = 0.2 + 1.1i$; (d) the intensity distribution $|\widetilde{q}|$ of equation (32) with $\lambda_1 = 0.1$, $\lambda_2 = 0.2$, $\lambda_3 = 0.3$, $F_1^{(1)} = 0.3+0.1i$, $F_1^{(2)} = 0.2 + 0.3i$, $F_2^{(2)} = 0.3 + 0.1i$, $F_2^{(1)} = 0.1 + 0.2i$, $F_3^{(1)} = 0.2 + 0.4i$, $F_3^{(2)} = 0.5 + 0.1i$.

$$\Delta_{12} = \begin{vmatrix} 1 & -\lambda_1^2 & N_1^{(2)} & \lambda_1 & \lambda_1 N_1^{(1)} & \lambda_1 N_1^{(2)} \\ 1 & -\lambda_2^2 & N_2^{(2)} & \lambda_2 & \lambda_2 N_2^{(1)} & \lambda_2 N_2^{(2)} \\ 1 & -\lambda_3^2 & N_3^{(2)} & \lambda_3 & \lambda_3 N_3^{(1)} & \lambda_3 N_3^{(2)} \\ 1 & -\lambda_4^2 & N_4^{(2)} & \lambda_4 & \lambda_4 N_4^{(1)} & \lambda_4 N_4^{(2)} \\ 1 & -\lambda_5^2 & N_5^{(2)} & \lambda_5 & \lambda_5 N_5^{(1)} & \lambda_5 N_5^{(2)} \\ 1 & -\lambda_6^2 & N_6^{(2)} & \lambda_6 & \lambda_6 N_6^{(1)} & \lambda_6 N_6^{(2)} \end{vmatrix},$$

$$\Delta_{13}=\begin{vmatrix}1 & N_1^{(1)} & -\lambda_1^2 & \lambda_1 & \lambda_1 N_1^{(1)} & \lambda_1 N_1^{(2)}\\ 1 & N_2^{(1)} & -\lambda_2^2 & \lambda_2 & \lambda_2 N_2^{(1)} & \lambda_2 N_2^{(2)}\\ 1 & N_3^{(1)} & -\lambda_3^2 & \lambda_3 & \lambda_3 N_3^{(1)} & \lambda_3 N_3^{(2)}\\ 1 & N_4^{(1)} & -\lambda_4^2 & \lambda_4 & \lambda_4 N_4^{(1)} & \lambda_4 N_4^{(2)}\\ 1 & N_5^{(1)} & -\lambda_5^2 & \lambda_5 & \lambda_5 N_5^{(1)} & \lambda_5 N_5^{(2)}\\ 1 & N_6^{(1)} & -\lambda_6^2 & \lambda_6 & \lambda_6 N_6^{(1)} & \lambda_6 N_6^{(2)}\end{vmatrix},$$

$$\Delta_{21}=\begin{vmatrix}-\lambda_1^2 N_1^{(1)} & N_1^{(1)} & N_1^{(2)} & \lambda_1 & \lambda_1 N_1^{(1)} & \lambda_1 N_1^{(2)}\\ -\lambda_2^2 N_2^{(1)} & N_2^{(1)} & N_2^{(2)} & \lambda_2 & \lambda_2 N_2^{(1)} & \lambda_2 N_2^{(2)}\\ -\lambda_3^2 N_3^{(1)} & N_3^{(1)} & N_3^{(2)} & \lambda_3 & \lambda_3 N_3^{(1)} & \lambda_3 N_3^{(2)}\\ -\lambda_4^2 N_4^{(1)} & N_4^{(1)} & N_4^{(2)} & \lambda_4 & \lambda_4 N_4^{(1)} & \lambda_4 N_4^{(2)}\\ -\lambda_5^2 N_5^{(1)} & N_5^{(1)} & N_5^{(2)} & \lambda_5 & \lambda_5 N_5^{(1)} & \lambda_5 N_5^{(2)}\\ -\lambda_6^2 N_6^{(1)} & N_6^{(1)} & N_6^{(2)} & \lambda_6 & \lambda_6 N_6^{(1)} & \lambda_6 N_6^{(2)}\end{vmatrix},$$

$$\Delta_{31}=\begin{vmatrix}-\lambda_1^2 N_1^{(2)} & N_1^{(1)} & N_1^{(2)} & \lambda_1 & \lambda_1 N_1^{(1)} & \lambda_1 N_1^{(2)}\\ -\lambda_2^2 N_2^{(2)} & N_2^{(1)} & N_2^{(2)} & \lambda_2 & \lambda_2 N_2^{(1)} & \lambda_2 N_2^{(2)}\\ -\lambda_3^2 N_3^{(2)} & N_3^{(1)} & N_3^{(2)} & \lambda_3 & \lambda_3 N_3^{(1)} & \lambda_3 N_3^{(2)}\\ -\lambda_4^2 N_4^{(2)} & N_4^{(1)} & N_4^{(2)} & \lambda_4 & \lambda_4 N_4^{(1)} & \lambda_4 N_4^{(2)}\\ -\lambda_5^2 N_5^{(2)} & N_5^{(1)} & N_5^{(2)} & \lambda_5 & \lambda_5 N_5^{(1)} & \lambda_5 N_5^{(2)}\\ -\lambda_6^2 N_6^{(2)} & N_6^{(1)} & N_6^{(2)} & \lambda_6 & \lambda_6 N_6^{(1)} & \lambda_6 N_6^{(2)}\end{vmatrix}.$$

Based on equations (9) and (27), we can obtain the following systems

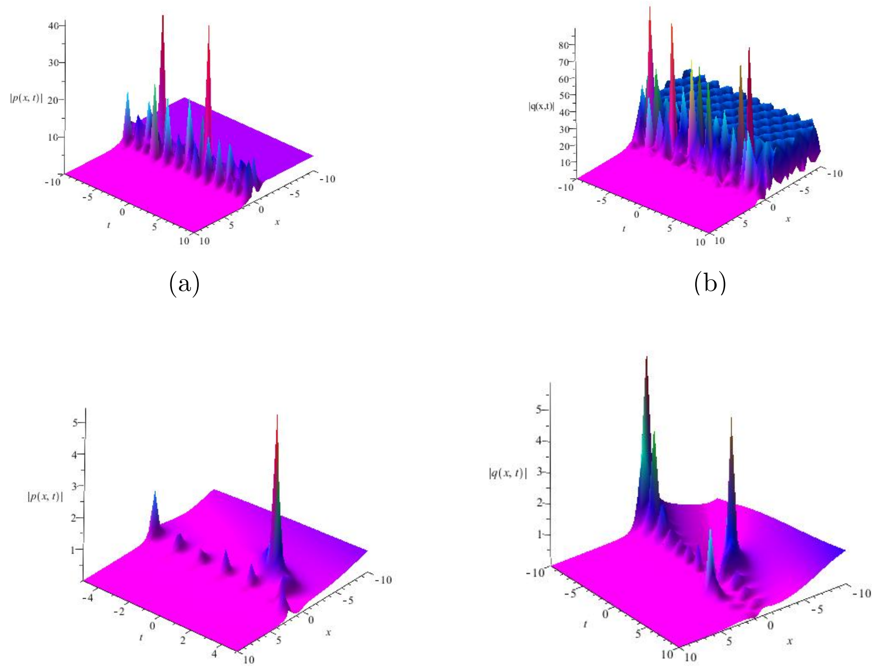
$$A_{12}[2]=\frac{\Delta_{12}}{\Delta},\quad A_{13}[2]=\frac{\Delta_{13}}{\Delta},\quad A_{21}[2]=\frac{\Delta_{21}}{\Delta},\quad A_{31}[2]=\frac{\Delta_{31}}{\Delta}, \tag{35}$$

where the analytic two-soliton solutions of CNNLS equation are obtained by the DT method as follows

$$\widetilde{p}[2]=-2i\frac{\Delta_{12}}{\Delta},\quad \widetilde{q}[2]=-2i\frac{\Delta_{13}}{\Delta}. \tag{36}$$

It is shown that solitary waves in nonautonomous nonlinear and dispersive systems can propagate in the form of so-called nonautonomous solitons or soliton-like similaritons. In Figure 2, the amplitude of the bright soliton also grows and decays with time. But the velocities before and after the peak time are different, which can be observed clearly from the nonsymmetric contour plot. The collapsing process after the largest amplitude is quicker, and vanishes rapidly. For illustration, the propagations and evolutions of $|\widetilde{p}[2]|$, $|\widetilde{q}[2]|$ are shown in Figure 2.

In equation (36), consider three values of $\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6$ are zero, or three values are same. In the first case, the numerator denominator is zero. In the second case, if the remaining three values are not identical, then the numerator is zero. Also there is the third case, that is there is one or two zeros, and the front six values $F_1^{(1)},F_1^{(2)},F_2^{(1)},F_2^{(2)},F_3^{(1)},F_3^{(2)},F_4^{(1)},F_4^{(2)},F_5^{(1)},F_5^{(2)},F_6^{(1)},F_6^{(2)}$ can not all be zero, with $F_4^{(2)}\neq F_6^{(2)}$, so that there is a solution.



(a) (b) (c) (d)

Figure 2: (a) The intensity distribution $|\widetilde{p}|$ of equation (36) with $\lambda_1 = 1$, $\lambda_2 = i$, $\lambda_3 = 0$, $\lambda_4 = i$, $\lambda_5 = 2i$, $\lambda_6 = 0$, $F_1^{(1)} = 0.3i$, $F_1^{(2)} = 0$, $F_2^{(1)} = 0.5i$, $F_2^{(2)} = 0$, $F_3^{(1)} = 0$, $F_3^{(2)} = 0.4i$, $F_4^{(1)} = 0$, $F_4^{(2)} = 0.1i$, $F_5^{(1)} = 0.2i$, $F_5^{(2)} = 0$, $F_6^{(1)} = 0$, $F_6^{(2)} = 2i$; (b) the intensity distribution $|\widetilde{q}|$ of equation (36) with $\lambda_1 = 1$, $\lambda_2 = i$, $\lambda_3 = 2$, $\lambda_4 = i$, $\lambda_5 = 2i$, $\lambda_6 = 1$, $F_1^{(1)} = i$, $F_1^{(2)} = 0$, $F_2^{(1)} = i$, $F_2^{(2)} = 1$, $F_3^{(1)} = 0$, $F_3^{(2)} = i$, $F_4^{(1)} = 2$, $F_4^{(2)} = i$, $F_5^{(1)} = i$, $F_5^{(2)} = 0$, $F_6^{(1)} = 1$, $F_6^{(2)} = 2i$; (c) the intensity distribution $|\widetilde{p}|$ of equation (36) with $\lambda_1 = 0.1$, $\lambda_2 = i$, $\lambda_3 = 0.2$, $\lambda_4 = i$, $\lambda_5 = 0.2i$, $\lambda_6 = 0.1$, $F_1^{(1)} = i$, $F_1^{(2)} = 0$, $F_2^{(1)} = i$, $F_2^{(2)} = 0.1$, $F_3^{(1)} = 0$, $F_3^{(2)} = i$, $F_4^{(1)} = 0.2$, $F_4^{(2)} = i$, $F_5^{(1)} = i$, $F_5^{(2)} = 0$, $F_6^{(1)} = 0.1$, $F_6^{(2)} = 0.2i$; (d) the intensity distribution $|\widetilde{q}|$ of equation (36) with $\lambda_1 = 0.1$, $\lambda_2 = i$, $\lambda_3 = 0.2$, $\lambda_4 = i$, $\lambda_5 = 0.2i$, $\lambda_6 = 0.1$, $F_1^{(1)} = i$, $F_1^{(2)} = 0$, $F_2^{(1)} = i$, $F_2^{(2)} = 0.1$, $F_3^{(1)} = 0$, $F_3^{(2)} = i$, $F_4^{(1)} = 0.2$, $F_4^{(2)} = i$, $F_5^{(1)} = i$, $F_5^{(2)} = 0$, $F_6^{(1)} = 0.1$, $F_6^{(2)} = 0.2i$.

In order to obtain the N-soliton solution formula of the CNNLS equation, we consider $N = n$ in equation (10), take $i = 0, 1, 2, \cdots, n-1$ and $j = 1, 2, \cdots, 3n$ into equation (11). We use the same way to get the form of N-soliton solution formula

$$\begin{cases} \sum\limits_{i=0}^{N-1} (A_{11}^{(i)} + A_{12}^{(i)} M_j^{(1)} + A_{13}^{(i)} M_j^{(2)}) \lambda_j^i = -\lambda_j^n, \\ \sum\limits_{i=0}^{N-1} (A_{21}^{(i)} + A_{22}^{(i)} M_j^{(1)} + A_{23}^{(i)} M_j^{(2)}) \lambda_j^i = -M_j^{(1)} \lambda_j^n, \\ \sum\limits_{i=0}^{N-1} (A_{31}^{(i)} + A_{32}^{(i)} M_j^{(1)} + A_{33}^{(i)} M_j^{(2)}) \lambda_j^i = -M_j^{(2)} \lambda_j^n, \end{cases} \tag{37}$$

and

$$\Delta = \begin{vmatrix} 1 & M_1^{(1)} & M_1^{(2)} & \lambda_1 & \cdots & \lambda_1^{n-1}M_1^{(1)} & \lambda_1^{n-1}M_1^{(2)} \\ 1 & M_2^{(1)} & M_2^{(2)} & \lambda_2 & \cdots & \lambda_2^{n-1}M_2^{(1)} & \lambda_2^{n-1}M_2^{(2)} \\ 1 & M_3^{(1)} & M_3^{(2)} & \lambda_3 & \cdots & \lambda_3^{n-1}M_3^{(1)} & \lambda_3^{n-1}M_3^{(2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & M_{3n}^{(1)} & M_{3n}^{(2)} & \lambda_{3n} & \cdots & \lambda_{3n}^{n-1}M_{3n}^{(1)} & \lambda_{3n}^{n-1}M_{3n}^{(2)} \end{vmatrix}.$$

We can obtain some solution of linear systems via the Cramer's rule

$$A_{12} = \frac{\Delta_{12}}{\Delta}, \quad A_{13} = \frac{\Delta_{13}}{\Delta}, \quad A_{21} = \frac{\Delta_{21}}{\Delta}, \quad A_{31} = \frac{\Delta_{31}}{\Delta}, \tag{38}$$

with

$$\Delta_{12} = \begin{vmatrix} 1 & -\lambda_1^n & M_1^{(2)} & \lambda_1 & \cdots & \lambda_1^{n-1}M_1^{(1)} & \lambda_1^{n-1}M_1^{(2)} \\ 1 & -\lambda_2^n & M_2^{(2)} & \lambda_2 & \cdots & \lambda_2^{n-1}M_2^{(1)} & \lambda_2^{n-1}M_2^{(2)} \\ 1 & -\lambda_3^n & M_3^{(2)} & \lambda_3 & \cdots & \lambda_3^{n-1}M_3^{(1)} & \lambda_3^{n-1}M_3^{(2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & -\lambda_{3n}^n & M_{3n}^{(2)} & \lambda_{3n} & \cdots & \lambda_{3n}^{n-1}M_{3n}^{(1)} & \lambda_{3n}^{n-1}M_{3n}^{(2)} \end{vmatrix},$$

$$\Delta_{13} = \begin{vmatrix} 1 & M_1^{(1)} & -\lambda_1^n & \lambda_1 & \cdots & \lambda_1^{n-1}M_1^{(1)} & \lambda_1^{n-1}M_1^{(2)} \\ 1 & M_2^{(1)} & -\lambda_2^n & \lambda_2 & \cdots & \lambda_2^{n-1}M_2^{(1)} & \lambda_2^{n-1}M_2^{(2)} \\ 1 & M_3^{(1)} & -\lambda_3^n & \lambda_3 & \cdots & \lambda_3^{n-1}M_3^{(1)} & \lambda_3^{n-1}M_3^{(2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & M_{3n}^{(1)} & -\lambda_{3n}^n & \lambda_{3n} & \cdots & \lambda_{3n}^{n-1}M_{3n}^{(1)} & \lambda_{3n}^{n-1}M_{3n}^{(2)} \end{vmatrix},$$

$$\Delta_{21} = \begin{vmatrix} -{\lambda_1}^n M_1^{(1)} & M_1^{(1)} & M_1^{(2)} & \lambda_1 & \cdots & \lambda_1^{n-1}M_1^{(1)} & \lambda_1^{n-1}M_1^{(2)} \\ -{\lambda_2}^n M_2^{(1)} & M_2^{(1)} & M_2^{(2)} & \lambda_2 & \cdots & \lambda_2^{n-1}M_2^{(1)} & \lambda_2^{n-1}M_2^{(2)} \\ -{\lambda_3}^n M_3^{(1)} & M_3^{(1)} & M_3^{(2)} & \lambda_3 & \cdots & \lambda_3^{n-1}M_3^{(1)} & \lambda_3^{n-1}M_3^{(2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -{\lambda_{3n}}^n M_{3n}^{(1)} & M_{3n}^{(1)} & M_{3n}^{(2)} & \lambda_{3n} & \cdots & \lambda_{3n}^{n-1}M_{3n}^{(1)} & \lambda_{3n}^{n-1}M_{3n}^{(2)} \end{vmatrix},$$

$$\Delta_{31} = \begin{vmatrix} -\lambda_1{}^n M_1^{(2)} & M_1^{(1)} & M_1^{(2)} & \lambda_1 & \cdots & \lambda_1^{n-1} M_1^{(1)} & \lambda_1^{n-1} M_1^{(2)} \\ -\lambda_2{}^n M_2^{(2)} & M_2^{(1)} & M_2^{(2)} & \lambda_2 & \cdots & \lambda_2^{n-1} M_2^{(1)} & \lambda_2^{n-1} M_2^{(2)} \\ -\lambda_3{}^n M_3^{(2)} & M_3^{(1)} & M_3^{(2)} & \lambda_3 & \cdots & \lambda_3^{n-1} M_3^{(1)} & \lambda_3^{n-1} M_3^{(2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\lambda_{3n}{}^n M_{3n}^{(2)} & M_{3n}^{(1)} & M_{3n}^{(2)} & \lambda_{3n} & \cdots & \lambda_{3n}^{n-1} M_{3n}^{(1)} & \lambda_{3n}^{n-1} M_{3n}^{(2)} \end{vmatrix},$$

where

$$M_j^{(1)} = \nu_j^{(1)} \mathrm{e}^{-i\lambda x + 2i\lambda^2 t}, \quad M_j^{(2)} = \nu_j^{(2)} \mathrm{e}^{-i\lambda x + 2i\lambda^2 t},$$

and $\nu_j^{(i)} = \mathrm{e}^{2iF_j^{(i)}}$ $(1 \leq i \leq 2,\ 1 \leq j \leq 3N)$.

Based on equations (9) and (27), we can obtain the following systems

$$A_{12}[N] = \frac{\Delta_{12}}{\Delta}, \quad A_{13}[N] = \frac{\Delta_{13}}{\Delta}, \quad A_{21}[N] = \frac{\Delta_{21}}{\Delta}, \quad A_{31}[N] = \frac{\Delta_{31}}{\Delta}, \tag{39}$$

where the analytic N-soliton solutions of CNNLS equation are obtained by the Darboux transformation method as follows

$$\widetilde{p}[N] = -2i\frac{\Delta_{12}}{\Delta}, \quad \widetilde{q}[N] = -2i\frac{\Delta_{13}}{\Delta}. \tag{40}$$

Figures 1 and 2 exhibit the exact one- and two-soliton solutions of equation (1), similar to the one-soliton solution, we obtain the N-soliton solution formula of the CNNLS equation with N-fold Darboux transformation. Based on the obtained solutions, the propagation and interaction structures of these multi-solitons are shown graphically: Figure 1 exhibits the evolution structures of the one-dark and one-dark solitons with $N = 1$; in Figure 2, the overtaking elastic interactions among the two-dark and two-bright solitons with $N = 2$. The results in this paper might be helpful for understanding some physical phenomena described in plasmas.

4 Conclusions

In this paper, we give the form of N-soliton solution of CNNLS equation with Darboux transformation. In addition, the one-soliton and two-soliton solutions are solved in detail. At present, few of the CNNLS equations are studied by the Darboux transformation. We obtain the one-, two- and N-soliton solution formulas of the CNNLS equation with N-fold Darboux transformation. Based on the obtained solutions, the propagation and interaction structures of these multi-solitons are shown, the evolution structures of the one-dark and one-bright solitons are exhibited with $N = 1$, and the overtaking elastic interactions among the two-dark and two-bright solitons are considered with $N = 2$. These results might be helpful for understanding physical phenomena in a nonlinear diffusion regularity of two nonlinear wave propagation in the medium, and can be applied widely in the field of nonlinear optics, which also plays an important role in meteorology.

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