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PERMANENCE OF A DISCRETE LOGISTIC EQUATION WITH PURE TIME DELAYS*†

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Abstract

In this paper we propose a discrete logistic system with pure delays. By giving the detail analysis of the right-hand side functional of the system, we consider its permanence property which is one of the most important topic in the study of population dynamics. The results obtained in this paper are good extensions of the existing results to the discrete case. Also we give an example to show the feasibility of our main results.

Keywords permanence; discrete; delay; logistic system
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1 Introduction

Recently, lots of scholars have investigated the logistic equation. For example, in [1], a sufficient condition was obtained for the existence and the attractivity of a unique almost periodic solution by constructing suitable Lyapunov functional and almost periodic functional hull theory. The authors [2] discussed the permanence and the global attractivity of the model, based on the boundedness of solutions of the corresponding autonomous logistic model. As we know, the traditional single species logistic equation takes the form:

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = N(t)\big(a - bN(t)\big). \tag{1.1}$$

Many scholars argued that a more suitable model should include some past state of the system, thus, a more suitable single species model should take the form:

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$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = N(t)\Big(a - bN(t) - cN(t-\tau) - d\int_0^\infty K(s)N(t-s)\mathrm{d}s\Big). \tag{1.2}$$

Here, a is a positive constant; b, c, d are all nonnegative constants and at least one of them is a positive constant. We divide system (1.2) into the following two cases:

(1) $b \equiv 0$, in this case, we call the system pure time delay system;

(2) $b \neq 0$, in this case, we call the system non-pure time delay system.

For the second case, by constructing a suitable Lyapunov function, one could easily show that condition $b > c + d$ is enough to ensure the existence of unique globally attractive positive equilibrium. However, the dynamic behaviors of the first case is very complicated. Indeed, for the infinite delay case, that is, $b = c = 0$ in system (1.2), under the assumption that $a(t)$ and $d(t)$ are all continuous ω-period functions, Gopalsamy [3] studied the following logistic system:

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = N(t)\Big[a(t) - d(t)\int_0^\infty K(s)N(t-s)\mathrm{d}s\Big], \tag{1.3}$$

and investigated the existence and uniqueness of the periodic solution of system (1.3). The main results of [3] is then generalized by Seifert [4] to the almost periodic case, but require a somewhat stronger assumption on the decay kernel $K(t)$, that is,

$$\int_0^\infty K(s)\mathrm{d}s = 1, \quad \int_0^\infty \mathrm{e}^{rs}K(s)\mathrm{d}s < \infty, \tag{1.4}$$

where r is some positive constant.

Feng [5] extended the main results of Seifert [4] to the following almost periodic system with delays:

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = N(t)\Big[a(t) - c(t)N(t-\tau) - d(t)\int_0^\infty K(s)N(t-s)\mathrm{d}s\Big]. \tag{1.5}$$

Feng also weakened condition (1.7) to the following condition:

$$\int_0^\infty K(s)\mathrm{d}s = 1, \quad \int_0^\infty s^2K(s)\mathrm{d}s < \infty. \tag{1.6}$$

Meng, Chen and Wang [6] further weakened condition (1.6) to the following condition:

$$\int_0^\infty K(s)\mathrm{d}s = 1, \quad \int_0^\infty sK(s)\mathrm{d}s < \infty. \tag{1.7}$$

By using almost periodic functional Hull theory and some computational techniques, Meng et al. [6] showed that condition (1.7) together with some other conditions is enough to ensure the boundedness and global asymptotically attractivity of system (1.5). For the case $b = d = 0$ in system (1.2), that is, for the system

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = N(t)\big(a - cN(t-\tau)\big),$$

there are still many open problem needed to be studied. For example, Open problem 4.1 in Kuang [7, page 171] is as follows: Is it true that if $\alpha < \frac{\pi}{2}$, then positive solutions of

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = \alpha N(t)\big[1 - N(t-1)\big]$$

tend to the steady state $x(t) \equiv 1$?

On the other hand, it has been found that the discrete time models governed by difference equations are more appropriate than the continuous ones when the populations have non-overlapping generations. Discrete time models can also provide efficient computational models of continuous models for numerical simulations (see [8-13]). However, to the best of the author's knowledge, to this day, there are still no scholars propose and study a discrete time analogue of system (1.5). This motivates us to propose the following discrete logistic system with pure delays:

$$x(n+1) = x(n)\exp\left\{a(n) - c(n)x(n-\tau) - b(n)\sum_{s=0}^{\infty}K(s)x(n-s)\right\}, \tag{1.8}$$

where $x(n)$ is the density of species at n-th generation; $a(n)$ denotes the birth rate; the terms $c(n)x(n-\tau)$ and $b(n)\sum\limits_{s=0}^{\infty}K(s)x(n-s)$ represent the negative feedback crowing and the effect of the past life history of the species on its present birth rate, respectively; τ is a positive integer; $K : [0,\infty) \to [0,\infty)$ is a bounded nonnegative sequence which satisfies

$$\sum_{s=0}^{\infty}K(s) = 1, \tag{1.9}$$

$a(n)$, $b(n)$ and $c(n)$ are all sequences which are bounded above and below by positive constants, more precisely, we have:

$$0 < a^L \le a^M < \infty, \quad 0 < b^L \le b^M < \infty, \quad 0 < c^L \le c^M < \infty. \tag{1.10}$$

Here, for any bounded sequence $\{f(n)\}$, we denote $f^M = \sup\limits_{n\in N}\{f(n)\}$ and $f^L = \inf\limits_{n\in N}\{f(n)\}$.

In this paper we will investigate the persistent property of the above system which is one of the most important topic in the study of population dynamics. System (1.8) is said to be permanent if there exist positive constants m and M which are independent of the solution of the system, such that any positive solution $x(n)$ satisfies $m \le \liminf\limits_{n\to+\infty} x(n) \le \limsup\limits_{n\to+\infty} x(n) \le M$. Also, by the biological meaning, we will focus our discussion on the positive solution of system (1.8). So it is

$$x(\theta) = \psi(\theta) \geq 0, \quad \theta = \cdots, -n, -n+1, \cdots, -1, 0; \quad x(0) > 0. \tag{1.11}$$

One could easily see that the solution of (1.8) with the initial condition (1.11) is well defined and remains positive for all $n \in N$.

This paper is organized as follows. The persistent property of system (1.8) is then studied in Section 2. An example in Section 3 shows the feasibility of our main results. In Section 4, we give a brief conclusion.

2 Main Results

In this section, we establish a permanent result for system (1.8).

Proposition 2.1 *There exists a positive constant M, which is independent of the solution of the system, such that*

$$\limsup_{n\to+\infty} x(n) \leq M := \frac{\exp(a^M - 1)}{\Big(c(n)\exp\Big\{-\sum_{i=n-\tau}^{n-1} a(i)\Big\}\Big)^L},$$

for any solution of system (1.8) *with positive initial values* (1.11).

Proof Let $x(n)$ be any positive solution of system (1.8), it follows that

$$x(n+1) \leq x(n)\exp\{a(n)\}. \tag{2.1}$$

So we have

$$x(n-\tau) \geq x(n)\exp\left\{-\sum_{i=n-\tau}^{n-1} a(i)\right\}. \tag{2.2}$$

From (1.1), we obtain that

$$x(n+1) \leq x(n)\exp\left\{a(n) - \left[c(n)\exp\left(-\sum_{i=n-\tau}^{n-1} a(i)\right)\right]x(n)\right\}. \tag{2.3}$$

Then we obtain

$$\limsup_{n\to\infty} x(n) \leq M, \tag{2.4}$$

where

$$M = \frac{\exp(a^M - 1)}{\Big(c(n)\exp\Big\{-\sum_{i=n-\tau}^{n-1} a(i)\Big\}\Big)^L}.$$

Thus we complete the proof of Proposition 2.1.

Proposition 2.2 *Any positive solution $x(n)$ of system* (1.8) *satisfies*

$$\limsup_{n\to+\infty} x(n) \geq \frac{a^L}{b^M + c^M}. \tag{2.5}$$

Proof Suppose that (2.5) is not true, then there exists a solution $x_0(n)$ of system (1.8) with positive initial values satisfying

$$\limsup_{n\to+\infty} x_0(n) = M_0 < \frac{a^L}{b^M + c^M}. \tag{2.6}$$

Let h be a small enough positive constant. From (2.6), there exists a positive constant N_1 such that

$$x_0(n) \le M_0 + \frac{h}{b^M + c^M} < \frac{a^L}{b^M + c^M} \quad \text{when } n > N_1.$$

Set $M_0' = \max_{n\in N} x_0(n)$. From (1.9), there exists a positive constant $N_2 > N_1$ such that

$$\sum_{s=N_2}^{+\infty} K(s) \le \frac{1}{M_0'}\Big(\frac{a^L - 2h}{b^M + c^M} - M_0\Big) \quad \text{when } n > N_2.$$

So

$$\sum_{s=0}^{+\infty} K(s)x_0(n-s) = \sum_{s=0}^{n-N_1} K(s)x_0(n-s) + \sum_{s=n-N_1}^{+\infty} K(s)x_0(n-s) \le \frac{a^L - h}{b^M + c^M}, \tag{2.7}$$

when $n \ge N_1 + N_2 + \tau$. Further,

$$x_0(n+1) \ge x_0(n)\exp\Big\{a^L - c^M \frac{a^L}{b^M + c^M} - b^M \frac{a^L - h}{b^M + c^M}\Big\} \ge \frac{b^M h}{b^M + c^M}.$$

As a result, we obtain that

$$x_0(n) \ge x_0(N_1 + N_2 + \tau)\exp\Big\{\frac{b^M h}{b^M + c^M}(n - N_1 - N_2 - \tau)\Big\},$$

which contracts (2.6). Thus we complete the proof of Proposition 2.2.

Proposition 2.3 *There exists a positive constant m such that any solution $x(n)$ of system* (1.8) *with positive initial values satisfies*

$$\liminf_{n\to+\infty} x(n) > m. \tag{2.8}$$

Proof Suppose that (2.8) is not true, then there exists a sequence $\{\phi_k(n)\}$, $k = 1, 2, \cdots$, such that

$$\liminf_{n\to+\infty} x_k(n) < \frac{1}{2k}, \quad x_k(n) := x(n, \phi_k(n)). \tag{2.9}$$

Choose a positive constant K_1 such that $\frac{1}{K_1} < \frac{a^L}{3(b^M + c^M)}$. Set $M_k' = \max_{n\in N} x_k(n)$ when $k \ge K_1$. From Proposition 2.1, there exists a positive constant N_k such that $x_k(n) \le M + 1$, $n \ge N_k$. On the other hand, by Proposition 2.2, we have

$$\limsup_{n\to+\infty} x_k(n) \ge \frac{a^L}{b^M + c^M}.$$

Hence, there exists a positive constant $\tau_k > N_k + \tau$ such that

$$x_k(\tau_k) \geq \frac{a^L}{3(b^M + c^M)} \quad \text{and} \quad \sum_{\tau_k - N_k}^{+\infty} \leq \frac{a^L}{3(b^M + c^M)M_k'}.$$

Also, there exists a $\sigma_k > \tau_k$ such that $x_k(\sigma_k) \leq \frac{1}{k}$ and

$$\frac{1}{k} < x_k(n) < \frac{a^L}{3(b^M + c^M)}, \quad n \in (\tau_k, \sigma_k). \tag{2.10}$$

Firstly we can prove that

$$\sigma_k - \tau_k \geq \frac{1}{(b^M + c^M)(M+1)} \ln \frac{a^L k}{3(b^M + c^M)}, \quad k \geq K_1. \tag{2.11}$$

In fact, when $k \geq K_1,\ n \in (\tau_k, \sigma_k)$,

$$\begin{aligned} \sum_{s=0}^{+\infty} K(s) x_k(n-s) &= \sum_{s=0}^{n-N_k} K(s) x_k(n-s) + \sum_{s=n-N_k}^{+\infty} K(s) x_k(n-s) \\ &\leq M + 1 + \frac{a^L}{b^M}. \end{aligned} \tag{2.12}$$

Further, we have

$$a(n) - c(n)x_k(n-\tau) - b(n) \sum_{s=0}^{+\infty} K(s) x_k(n-s) \geq -(b^M + c^M)(M_0 + 1).$$

As a result, $x_k(n+1) \geq x_k(n) \exp\{-(b^M + c^M)(M+1)\}$. Then we obtain that

$$\frac{1}{k} \geq \frac{a^L}{3(b^M + c^M)} \exp\{-(b^M + c^M)(M+1)(\sigma_k - \tau_k)\},$$

so (2.11) is true.

From (1.9), there exists a positive constant N such that

$$\sum_{s=N}^{+\infty} K(s) < \frac{a^L}{3(b^M + c^M)(M+1)}.$$

Set $K > K_1$, $\frac{1}{(b^M+c^M)(M+1)} \ln \frac{a^L K}{3(b^M+c^M)} > N$ and $\tau_K + N < \sigma_K$, then

$$\begin{aligned} &\sum_{s=0}^{+\infty} K(s) x_K(n-s) \\ = &\sum_{s=0}^{n-\tau_K} K(s) x_K(n-s) + \sum_{s=n-\tau_K}^{n-N_K} K(s) x_K(n-s) + \sum_{s=n-N_K}^{+\infty} K(s) x_K(n-s) \\ \leq &\frac{a^L}{3(b^M + c^M)} \sum_{s=0}^{n-\tau_K} K(s) + (M+1) \sum_{s=n-\tau_K}^{n-N_K} K(s) + M_K' \sum_{s=n-N_K}^{+\infty} K(s) \end{aligned}$$

$$\leq \frac{a^L}{3(b^M+c^M)} + (M+1)\sum_{s=N}^{+\infty} K(s) + M'_K \sum_{s=\tau_K-N_K}^{+\infty} K(s)$$
$$< \frac{a^L}{3(b^M+c^M)} + (M+1)\frac{a^L}{3(b^M+c^M)(M+1)} + M'_K \frac{a^L}{3(b^M+c^M)M'_K}$$
$$= \frac{a^L}{b^M+c^M}, \tag{2.13}$$

when $n \in [\tau_k + N, \sigma_k]$. So

$$x(n+1) \geq x(n)\exp\left\{a^L - c^M \frac{a^L}{3(b^M+c^M)} - b^M \frac{a^L}{(b^M+c^M)}\right\} > 0, \quad n \in [\tau_K + N, \sigma_k],$$

which follows that $x_K(n) > x_k(\tau_K + N) > \frac{1}{K}$. This contracts $x_K(\sigma_K) = \frac{1}{K}$, so (2.8) is true. Thus we complete the proof of Proposition 2.3.

From Propositions 2.1, 2.2 and 2.3, we easily have the following theorem:

Theorem 2.1 *System* (1.8) *with initial conditions* (1.11) *is permanent.*

3 Numerical Simulation

Consider system (1.8) with the following coefficients

$$a(n) = 1.1 + 0.4\cos n, \quad c(n) = 0.6 + 0.2\cos n,$$
$$b(n) = 1.1, \quad \tau = 2, \quad K(s) = \frac{\mathrm{e}-1}{\mathrm{e}}\mathrm{e}^{-s}.$$

System (1.8) is permanent. Our numerical simulation supports our result (see Figure 1).

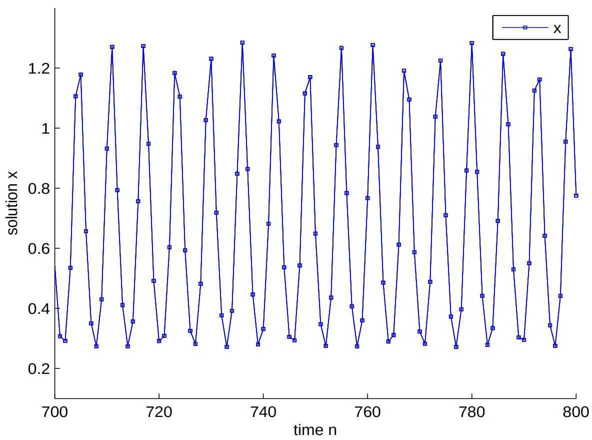

Figure 1: Dynamics behaviors of system (1.8) with initial condition $x(\theta) \equiv 0.63, 0.67, 0.66, \ \theta \in (-\infty, 0]$.

4 Conclusion

In this paper we have investigated the permanence property for a discrete logistic system with pure delays. The results obtained in this paper have been a good extension of the corresponding results obtained in Feng [5] to the discrete case. We also have presented an example to verify our main results, which show that delay does not influence the permanence property of the discrete system.

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