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STUDY OF THE STABILITY BEHAVIOUR AND THE BOUNDEDNESS OF SOLUTIONS TO A CERTAIN THIRD-ORDER DIFFERENTIAL EQUATION WITH A RETARDED ARGUMENT*

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Abstract

Lyapunov direct method is employed to investigate the asymptotic behaviour and the boundedness of solutions to a certain third-order differential equation with delay and some new results are obtained. Our results improve and complement some earlier results. Two examples are given to illustrate the importance of the topic and the main results obtained.

Keywords delay differential equations; asymptotic behaviour; stability; third-order differential equations; Lyapunov functional

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1 Introduction

Differential equations (DEs) are used as tools for mathematical modeling in many fields of life science. When a model does not incorporate a dependence on its past history, it generally consists of so-called ordinary differential equation (ODEs). Models incorporating past history generally include delay differential equations (DDEs) or functional differential equations (FDEs). In applications, the future behaviour of many phenomena is assumed to be described by the solutions of an DDEs, which implies that the future behaviour is uniquely determined by the present and independent phenomena of the past. In FDEs, the past exerts its influence in a significant manner upon the future. Many phenomena are more suitable to be described by DDEs than ODEs. In many processes including physical, chemical, political, economical, biological, and control systems, time-delay is an important factor. In particular the third-order delay differential equations usually describe the phenomena in various areas of applied mathematics and physics, for instance deflection of

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bucking beam with a variable cross-section, electromagnetic waves, gravity driven flows, etc.

As we know the study of qualitative properties of solutions, such as stability and boundedness, is very important in the theory of differential equations. Since it is difficult to solve solutions to DEs, Lyapunov method is usually used to study the stability and boundedness of the equations.

Many good results have been obtained on the qualitative behaviour of solutions to some kinds of third-order DDEs by Zhu [23], Sadek [14–16], Abou-El-Ela et al. [1], Tunç [18–22], Ademola et al. [2,3], Afuwape and Omeike [4], Bai and Guo [5], Shekhare et al. [17], Remili et al. [11], and the references therein.

Numerous authors have obtained some very interesting results about the asymptotic behaviour of solutions to third-order DDEs, for example, Chen and Guan [7], Mahmoud [8], Remili et al. [10, 12, 13], etc.

In 2016, Remili and Oudjedi [12] studied the ultimate boundedness and the asymptotic behaviour of solutions to a third-order nonlinear DDE of the form

$$[\Omega(x,x')x'']'+(f(x,x')x')'+g(x(t-r(t)),x'(t-r(t)))+h(x(t-r(t)))=p(t,x,x',x''),$$

where f, g, h, Ω and p are continuous functions in their respective arguments with $g(x, 0) = h(0) = 0$.

In 2017, Remili et al. [13] investigated the stability and ultimate boundedness of solutions to a kind of third-order DDE as follows

$$[g(x''(t))x''(t)]' + (h(x'(t))x'(t))' + (\phi(x(t))x(t))' + f(x(t-r)) = e(t),$$

where $r > 0$ is a fixed delay; e, f, g, h and ϕ are continuous functions in their respective arguments with $f(0) = 0$.

The main objective of this research is to study the asymptotic stability and the boundedness of solutions to a nonlinear third-order DDE

$$\begin{aligned}&[h(x(t))x''(t)]' + [p(x(t))x'(t)]' + g(x'(t-r(t))) + f(x(t-r(t)))\\ =\ & e(t, x(t), x'(t), x''(t)),\end{aligned} \tag{1.1}$$

where h, p, g, f and e are continuous functions with $g(0) = f(0) = 0$, and the derivatives $h'(u) = \frac{\mathrm{d}h}{\mathrm{d}u}$ and $p'(u) = \frac{\mathrm{d}p}{\mathrm{d}u}$ exist and are also continuous.

We can take

$$h'(x(t))x'(t) = \theta_1, \quad p'(x(t))x'(t) = \theta_2. \tag{1.2}$$

Remark In equation (1.1), if $h(x(t)) = 1$ and $p(x(t)) = a$, then equation (1.1) is reduced to the equation in Sadek [14].

2 Stability Result

To prove the stability result, we shall give some important theorems about the stability of solutions to DDEs.

Consider the general autonomous DDE

$$\dot{x}(t) = f(x_t), \quad x_t(\theta) = x(t+\theta), \quad -r \le \theta \le 0,\ t \ge 0, \tag{2.1}$$

where $f : C_H \to \mathbb{R}^n$ is a continuous mapping, $f(0) = 0$. We suppose that f maps closed bounded sets into bounded sets of $\mathbb{R}^n$. Here $(C, \|\cdot\|)$ is the Banach space of continuous functions $\phi : [-r, 0] \to \mathbb{R}^n$ with supremum norm, $r > 0$; C_H is an open ball of radius H in C; $C_H := \{\phi \in (C[-r,0], \mathbb{R}^n) : \|\phi\| < H\}$.

Theorem 2.1[6] *Let $V : C_H \to \mathbb{R}$ be a continuous functional satisfying a local Lipschitz condition, $V(0) = 0$, such that*

(i) $W_1(|\phi(0)|) \le V(\phi) \le W_2(\|\phi\|)$, *where $W_1(r)$ and $W_2(r)$ are wedges;*

(ii) $\dot{V}_{(2.1)}(\phi) \le 0$, *for $\phi \in C_H$.*

Then the zero solution to (2.1) *is uniformly stable.*

Theorem 2.2[6] *If there are a Lyapunov functional for* (2.1) *and wedges W_i $(i = 1, 2, 3)$, such that*

(i) $W_1(|\phi(0)|) \le V(\phi) \le W_2(\|\phi\|)$;

(ii) $\dot{V}_{(2.1)}(\phi) \le -W_3(|\phi|)$.

Then the zero solution to (2.1) *is uniformly asymptotically stable.*

Now, we shall give the main theorem and its proof.

Theorem 2.3 *Suppose that there are positive constants a_0, a_1, a_2, a_3, b_1, b_2, L_1, L_2, γ and β which satisfy the following conditions:*

(i) $|f'(x)| \le L_1,\ \sup\{f'(x)\} = b_1,\ f(x)\mathrm{sgn}x > 0$, *for $x \ne 0$;*

(ii) $g(0) = 0,\ |g'(y)| \le L_2,\ \frac{g(y)}{y} \ge b_2,\ y \ne 0$;

(iii) $0 < a_2 \le h(x) \le a_0,\ a_1 \le p(x) \le a_3$;

(iv) $0 \le r(t) \le \gamma,\ r'(t) \le \beta,\ 0 < \beta < 1$;

(v) $a_1 b_2 - a_2 b_1 > 0,\ a_1 - a_0\mu > 0$;

(vi) $\int_{-\infty}^{\infty} [|h'(u)| + |p'(u)|] \mathrm{d}u < \infty$,

provided that

$$\gamma < \min\left\{ \frac{(1-\beta)(a_1b_2-a_2b_1)}{2\mu(L_1+L_2)(1-\beta)+L_1(1+a_2\mu)}, \frac{a_2(1-\beta)(a_1b_2-a_2b_1)}{2b_2\big\{L_2\{1+\mu a_2+a_2^2(1-\beta)\}+L_1a_2^2(1-\beta)\big\}} \right\},$$

where

$$\mu = \frac{a_1b_2 + a_2b_1}{2a_2b_2}.$$

Then the zero solution to (1.1) *with $e = 0$ is uniformly asymptotically stable.*

Proof When $e = 0$, equation (1.1) is equivalent to

$$
\begin{aligned}
&x' = y, \quad y' = \frac{z}{h(x)},\\
&z' = -p(x)\frac{z}{h(x)} - \theta_2 y - f(x) - g(y)\\
&\quad + \int_{t-r(t)}^{t} f'(x(s))y(s)\mathrm{d}s + \int_{t-r(t)}^{t} g'(y(s))\frac{z(s)}{h(x(s))}\mathrm{d}s. \qquad (2.2)
\end{aligned}
$$

The Lyapunov functional of the above system could be defined as:

$$
\begin{aligned}
V(x_t, y_t, z_t) = {}& \mu\int_0^x f(\xi)\mathrm{d}\xi + f(x)y + \frac{\mu}{2}p(x)y^2 + \int_0^y g(\eta)\mathrm{d}\eta + \mu yz + \frac{1}{2h(x)}z^2\\
&+\lambda\int_{-r(t)}^{0}\int_{t+s}^{t} y^2(\theta)\mathrm{d}\theta\mathrm{d}s + \delta\int_{-r(t)}^{0}\int_{t+s}^{t} z^2(\theta)\mathrm{d}\theta\mathrm{d}s. \qquad (2.3)
\end{aligned}
$$

From conditions (i)-(iii) and using the mean-value theorem, we get

$$
\begin{aligned}
V(x_t, y_t, z_t) \le {}& \mu\int_0^x L_1\xi\mathrm{d}\xi + L_1 xy + \frac{\mu a_3}{2}y^2 + \int_0^y L_2\eta\mathrm{d}\eta + \mu yz + \frac{1}{2a_2}z^2\\
&+\lambda\int_{t-r(t)}^{t}(\theta - t + r(t))y^2(\theta)\mathrm{d}\theta + \delta\int_{t-r(t)}^{t}(\theta - t + r(t))z^2(\theta)\mathrm{d}\theta.
\end{aligned}
$$

According to the inequality $xy \le \frac{1}{2}(x^2+y^2)$, we find

$$
\begin{aligned}
V(x_t, y_t, z_t) \le {}& \frac{\mu L_1}{2}x^2 + \frac{L_1}{2}x^2 + \frac{L_1}{2}y^2 + \frac{\mu a_3}{2}y^2 + \frac{L_2}{2}y^2 + \frac{\mu}{2}y^2 + \frac{\mu}{2}z^2 + \frac{1}{2a_2}z^2\\
&+\frac{\lambda}{2}r^2(t)\|y\|^2 + \frac{\delta}{2}r^2(t)\|z\|^2.
\end{aligned}
$$

Considering $r(t) \le \gamma$ in (iv), we obtain

$$
\begin{aligned}
V(x_t, y_t, z_t) \le {}& \frac{(\mu+1)L_1}{2}\|x\|^2 + \frac{1}{2}\{L_1 + L_2 + \mu a_3 + \mu + \lambda\gamma^2\}\|y\|^2\\
&+\frac{1}{2}\Big\{\mu + \frac{1}{a_2} + \delta\gamma^2\Big\}\|z\|^2.
\end{aligned}
$$

Then there exists a positive constant D_0 such as

$$
V(x_t, y_t, z_t) \le D_0(x^2 + y^2 + z^2). \qquad (2.4)
$$

Since $\int_{-r(t)}^{0}\int_{t+s}^{t} y^2(\theta)\mathrm{d}\theta\mathrm{d}s$ and $\int_{-r(t)}^{0}\int_{t+s}^{t} z^2(\theta)\mathrm{d}\theta\mathrm{d}s$ are non-negative, from conditions (i)-(iii) of Theorem 2.3, we find

$$
V(x_t, y_t, z_t) \ge \mu\int_0^x f(\xi)\mathrm{d}\xi + f(x)y + \frac{\mu a_1}{2}y^2 + \frac{b_2}{2}y^2 + \mu yz + \frac{1}{2a_0}z^2.
$$

It follows that

$$
V(x_t, y_t, z_t) \ge \frac{1}{2b_2}(b_2 y + f(x))^2 + \mu\int_0^x f(\xi)\mathrm{d}\xi + \frac{\mu a_1}{2}y^2 - \frac{1}{2b_2}(f(x))^2 + \mu yz + \frac{1}{2a_0}z^2.
$$

Then we have

$$V(x_t,y_t,z_t)\geq \frac{1}{2b_2}(b_2y+f(x))^2+\frac{1}{2b_2y^2}\left[4\int_0^x f(\xi)\left(\int_0^y(\mu b_2-f'(\xi))\eta \mathrm{d}\eta\right)\mathrm{d}\xi\right]$$
$$+\frac{a_0}{2}\left(\mu y+\frac{z}{a_0}\right)^2+\frac{\mu}{2}(a_1-a_0\mu)y^2.$$

Since $f(x)\mathrm{sgn}x>0$, $a_1-a_0\mu>0$, $\mu b_2-f'(\xi)\geq \mu b_2-b_1=\frac{a_1b_2-a_2b_1}{2a_2}>0$, we find

$$V(x_t,y_t,z_t)\geq \frac{1}{2b_2}(b_2y+f(x))^2+\frac{a_0}{2}\left(\mu y+\frac{z}{a_0}\right)^2.$$

There exists a positive constant D_1 such that

$$V(x_t,y_t,z_t)\geq D_1(x^2+y^2+z^2). \tag{2.5}$$

Now, differentiating both sides of (2.3) along the solution to system (2.2) and from (1.2), we have

$$\frac{\mathrm{d}V}{\mathrm{d}t}=f'(x)y^2-\frac{\mu\theta_2}{2}y^2+\mu\frac{z^2}{h(x)}-\mu y g(y)-\frac{p(x)}{(h(x))^2}z^2-\frac{\theta_1}{2(h(x))^2}z^2-\frac{\theta_2}{h(x)}yz$$
$$+\left(\mu y+\frac{z}{h(x)}\right)\int_{t-r(t)}^t f'(x(s))y(s)\mathrm{d}s+\left(\mu y+\frac{z}{h(x)}\right)\int_{t-r(t)}^t g'(y(s))\frac{z(s)}{h(x(s))}\mathrm{d}s$$
$$+\lambda y^2r(t)-\lambda(1-r'(t))\int_{t-r(t)}^t y^2(\theta)\mathrm{d}\theta+\delta z^2r(t)-\delta(1-r'(t))\int_{t-r(t)}^t z^2(\theta)\mathrm{d}\theta.$$

By conditions (i)-(iii), we obtain

$$\frac{\mathrm{d}V}{\mathrm{d}t}\leq b_1y^2-\frac{\mu\theta_2}{2}y^2+\frac{\mu}{a_2}z^2-\mu b_2y^2-\frac{\theta_1}{2a_0^2}z^2-\frac{a_1}{a_2^2}z^2-\frac{\theta_2}{a_0}yz$$
$$+\left(\mu y+\frac{z}{a_2}\right)\int_{t-r(t)}^t f'(x(s))y(s)\mathrm{d}s+\left(\mu y+\frac{z}{a_2}\right)\int_{t-r(t)}^t g'(y(s))\frac{z(s)}{h(x(s))}\mathrm{d}s$$
$$+\lambda y^2r(t)+\delta z^2r(t)-\lambda(1-r'(t))\int_{t-r(t)}^t y^2(\theta)\mathrm{d}\theta-\delta(1-r'(t))\int_{t-r(t)}^t z^2(\theta)\mathrm{d}\theta.$$

By conditions $|f'(x)|\leq L_1$ and $|g'(y)|\leq L_2$, using the inequality $xy\leq\frac{1}{2}(x^2+y^2)$, we conclude

$$\frac{\mathrm{d}V}{\mathrm{d}t}\leq b_1y^2-\frac{\mu\theta_2}{2}y^2+\frac{\mu}{a_2}z^2-\mu b_2y^2-\frac{\theta_1}{2a_0^2}z^2-\frac{a_1}{a_2^2}z^2+\frac{\theta_2}{2a_0}y^2+\frac{\theta_2}{2a_0}z^2$$
$$+\frac{\mu L_1}{2}r(t)y^2+\frac{\mu L_1}{2}\int_{t-r(t)}^t y^2(s)\mathrm{d}s+\frac{L_1}{2a_2}r(t)z^2+\frac{L_1}{2a_2}\int_{t-r(t)}^t y^2(s)\mathrm{d}s$$
$$+\frac{\mu L_2}{2}r(t)y^2+\frac{\mu L_2}{2a_2^2}\int_{t-r(t)}^t z^2(s)\mathrm{d}s+\frac{L_2}{2a_2}r(t)z^2+\frac{L_2}{2a_2^3}\int_{t-r(t)}^t z^2(s)\mathrm{d}s$$
$$+\lambda y^2r(t)+\delta z^2r(t)-\lambda(1-r'(t))\int_{t-r(t)}^t y^2(\theta)\mathrm{d}\theta-\delta(1-r'(t))\int_{t-r(t)}^t z^2(\theta)\mathrm{d}\theta.$$

It follows from condition (iv) that

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t}\le&-\Big\{\mu b_2-b_1+\frac{\mu\theta_2}{2}-\frac{\theta_2}{2a_0}-\frac{\mu L_1}{2}\gamma-\frac{\mu L_2}{2}\gamma-\lambda\gamma\Big\}y^2\\&-\Big\{\frac{a_1}{a_2^2}-\frac{\mu}{a_2}+\frac{\theta_1}{2a_0^2}-\frac{\theta_2}{2a_0}-\frac{L_1}{2a_2}\gamma-\frac{L_2}{2a_2^2}\gamma-\delta\gamma\Big\}z^2\\&+\Big\{\frac{\mu}{2}L_1+\frac{L_1}{2a_2}-\lambda(1-\beta)\Big\}\int_{t-r(t)}^t y^2(\theta)\mathrm{d}\theta\\&+\Big\{\frac{\mu}{2a_2^2}L_2+\frac{L_2}{2a_2^3}-\delta(1-\beta)\Big\}\int_{t-r(t)}^t z^2(\theta)\mathrm{d}\theta.\end{aligned}$$

Take

$$\lambda=\Big(\frac{\mu}{2}L_1+\frac{L_1}{2a_2}\Big)\frac{1}{1-\beta}\quad\text{and}\quad\delta=\Big(\frac{\mu}{2a_2^2}L_2+\frac{L_2}{2a_2^3}\Big)\frac{1}{1-\beta},$$

so the above equation becomes

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t}\le&-\Big\{\frac{a_1b_2-a_2b_1}{2a_2}-\frac{\mu(L_1+L_2)(1-\beta)+L_1(1+a_2\mu)}{2a_2(1-\beta)}\gamma\Big\}y^2\\&-\Big\{\frac{a_1b_2-b_1a_2}{2a_2^2b_2}-\frac{L_2\{\mu a_2+a_2^2(1-\beta)+1\}+L_1a_2^2(1-\beta)}{2a_2^3(1-\beta)}\gamma\Big\}z^2+R(t),\end{aligned}$$

where

$$\begin{aligned}R(t)&=-\frac{\theta_2}{a_0}yz-\frac{\mu\theta_2}{2}y^2-\frac{\theta_1}{2a_0^2}z^2\\&\le\frac{1}{2a_0}|\theta_2||y^2+z^2|+\frac{\mu}{2}|\theta_2|y^2+\frac{|\theta_1|}{2a_0^2}z^2\\&\le\Big\{\frac{1}{2a_0^2}|\theta_1|+\Big(\frac{1}{2a_0}+\frac{\mu}{2}\Big)|\theta_2|\Big\}|y^2+z^2|.\end{aligned}$$

Thus, we find

$$R(t)\le D_2(|\theta_1|+|\theta_2|)(y^2+z^2),$$

where

$$D_2=\frac{1}{2a_0^2}\Big(1+a_0+\frac{a_0^2}{2}\mu\Big).$$

From inequality (2.5), taking $G(t)=|\theta_1|+|\theta_2|$, we obtain

$$R(t)\le D_2G(t)\frac{V}{D_1}.$$

Then

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t}\le&-\Big\{\frac{a_1b_2-a_2b_1}{2a_2}-\frac{\mu(L_1+L_2)(1-\beta)+L_1(1+a_2\mu)}{2a_2(1-\beta)}\gamma\Big\}y^2\\&-\Big\{\frac{a_1b_2-a_2b_1}{2a_2^2b_2}-\frac{L_2\{\mu a_2+a_2^2(1-\beta)+1\}+L_1a_2^2(1-\beta)}{2a_2^3(1-\beta)}\gamma\Big\}z^2+\frac{D_2}{D_1}G(t)V.\end{aligned}$$

So we can write the time derivative of Lyapunov functional $V(x_t, y_t, z_t)$ as

$$\frac{\mathrm{d}V}{\mathrm{d}t} \leq -D_3 y^2 - D_4 z^2 + \frac{D_2}{D_1} G(t) V, \tag{2.6}$$

provided that

$$\gamma < \min\Big\{\frac{(1-\beta)(a_1 b_2 - a_2 b_1)}{2\mu(L_1+L_2)(1-\beta)+L_1(1+a_2\mu)}, \frac{a_2(1-\beta)(a_1 b_2 - a_2 b_1)}{2b_2\{L_2\{1+\mu a_2 + a_2^2(1-\beta)\}+L_1 a_2^2(1-\beta)\}}\Big\}.$$

Now, we define the continuously differentiable functional W as follows

$$W = \exp\Big(-\frac{w(t)}{\eta}\Big)V,$$

where

$$w(t) = \int_0^t G(s)\mathrm{d}s \leq \int_{\alpha_1}^{\alpha_2} [|h'(u)| + |p'(u)|]\mathrm{d}s \leq \int_{-\infty}^{\infty} [|h'(u)| + |p'(u)|]\mathrm{d}u \leq N < \infty,$$

with $\alpha_1 = \min\{x(0), x(t)\}$ and $\alpha_2 = \max\{x(0), x(t)\}$.

Therefore, if we take $\eta = \frac{D_1}{D_2}$, then

$$W = \exp\Big(-\frac{w(t)}{\eta}\Big)V = \exp\Big(-\frac{D_2}{D_1}w(t)\Big)V.$$

Taking the derivative of this equation, from (2.6) there is

$$W' \leq \exp\Big(-\frac{D_2}{D_1}w(t)\Big)(-D_3 y^2 - D_4 z^2) \leq -\alpha(y^2 + z^2), \tag{2.7}$$

where $\alpha = \exp\big(-\frac{D_2}{D_1}w(t)\big)\min\{D_3, D_4\} > 0$.

Hence, from (2.7), $W_3(\|X\|) = \alpha(y^2+z^2)$ is a positive definite function and from inequalities (2.4), (2.5), the Lyapunov functional $V(x_t, y_t, z_t)$ satisfies all conditions of Theorem 2.2.

Therefore we conclude that the zero solution to equation (1.1) is uniformly asymptotically stable.

Thus the proof of Theorem 2.3 is now finished.

3 Boundedness of Solutions

In this case $e(t) \neq 0$, equation (1.1) is equivalent to the following system

$$\begin{aligned} x' &= y, \quad y' = \frac{z}{h(x)}, \\ z' &= -p(x)\frac{z}{h(x)} - \theta_2 y - f(x) - g(y) + \int_{t-r(t)}^{t} f'(x(s))y(s)\mathrm{d}s \\ &\quad + \int_{t-r(t)}^{t} g'(y(s))\frac{z(s)}{h(x(s))}\mathrm{d}s + e(t, x(t), y(t), z(t)). \end{aligned} \tag{3.1}$$

We can obtain the following theorem.

Theorem 3.1 *In addition to conditions* (i)-(vi), *we assume that*

$$|e(t,x(t),y(t),z(t))| \le q(t),$$

where $\max(q(t)) < \infty$ *and* $q(t) \in L^1(0,\infty)$ *with* $L^1(0,\infty)$ *being a space of integrable Lebesgue function. Then there exists a finite positive constant* D, *such that the solution* $x(t)$ *defined by the initial functions*

$$x(t) = \psi(t), \quad x'(t) = \psi'(t), \quad x''(t) = \psi''(t)$$

satisfies the inequalities

$$|x(t)| < D, \quad |x'(t)| < D, \quad |x''(t)| < D, \quad \textit{for any } t \ge t_0.$$

Proof By the conditions of Theorem 3.1, and inequality (2.6) for system (3.1), we can write

$$\frac{\mathrm{d}V}{\mathrm{d}t} \le -D_3y^2 - D_4z^2 + \frac{D_2}{D_1}G(t)V + \Big(\mu|y| + \frac{1}{a_2}|z|\Big)|e(t,x(t),y(t),z(t))|.$$

Accordingly, the last inequality becomes

$$\frac{\mathrm{d}V}{\mathrm{d}t} \le \frac{D_2}{D_1}G(t)V + \Big(\mu|y| + \frac{1}{a_2}|z|\Big)q(t).$$

If we recall the inequalities $|y| < 1 + y^2$ and $|z| < 1 + z^2$, then we get

$$\frac{\mathrm{d}V}{\mathrm{d}t} \le k_1G(t)V + k_2(2 + y^2 + z^2)q(t),$$

where $k_1 = \frac{D_2}{D_1}$, $k_2 = \max\{\mu, \frac{1}{a_2}\}$. From (2.5), we have $y^2 + z^2 \le D_1^{-1}V(x_t, y_t, z_t)$, then

$$\frac{\mathrm{d}V}{\mathrm{d}t} \le 2k_2q(t) + [k_1G(t) + k_2D_1^{-1}q(t)]V \le 2k_2q(t) + k_3[G(t) + q(t)]V,$$

where $k_3 = \max\{k_1, k_2D_1^{-1}\}$. Integrating the previous inequality from 0 to t, noting $G(t), q(t) \in L^1(0,\infty)$ and using the Gronwall-Reid-Bellman inequality, we conclude

$$V(x_t, y_t, z_t) \le \Big[V(x_0, y_0, z_0) + 2k_2\int_0^t q(s)\mathrm{d}s\Big]\exp\Big(k_3\int_0^t T(s)\mathrm{d}s\Big) = c < \infty, \quad (3.2)$$

for a positive constant c. From inequalities (2.5) and (3.2), we obtain $x^2 + y^2 + z^2 \le D_1^{-1}V \le k_4$, then we conclude

$$|x(t)| \le k_4, \quad |x'(t)| = |y(t)| \le k_4, \quad |x''(t)| = \Big|\frac{z}{h(x)}\Big| \le \frac{k_4}{a_2}.$$

Thus, we have

$$|x(t)| \le D, \quad |x'(t)| \le D, \quad |x''(t)| \le D, \quad D = \max\Big\{k_4, \frac{k_4}{a_2}\Big\}, \quad \text{for all } t \ge t_0.$$

The proof of Theorem 3.1 is now finished.

4 Examples

Example 4.1 In this example we shall study the stability of a third-order nonlinear DDE of the following form

$$\Big[\Big(\frac{e^x}{1+e^{2x}}+\frac{3}{2}\Big)x''(t)\Big]'+\Big[\Big(\frac{2}{3}+e^{-x^2}\Big)x'(t)\Big]'+\Big[36\,x'(t-r(t))+\frac{x'(t-r(t))}{1+|x'(t-r(t))|}\Big]$$
$$+\frac{3}{2}x(t-r(t))+\frac{1}{2}\sin\left(x(t-r(t))\right)=0. \tag{4.1}$$

It is obvious that

$$\frac{3}{2}\le\frac{e^x}{1+e^{2x}}+\frac{3}{2}\le 2$$

and

$$\int_{-\infty}^{\infty}|h'(x(u))|\mathrm{d}u\le\int_{-\infty}^{\infty}\Big[\Big|\frac{e^u}{(1+e^{2u})^2}\Big|+\Big|\frac{e^{3u}}{(1+e^{2u})^2}\Big|\Big]\mathrm{d}u.$$

Then, it follows that

$$\int_{-\infty}^{\infty}|h'(x(u))|\mathrm{d}u\le\frac{\pi}{4}+\frac{\pi}{4}=\frac{\pi}{2}<\infty.$$

Also, there are

$$\frac{2}{3}\le\frac{2}{3}+e^{-x^2}\le\frac{5}{3}$$

and

$$\int_{-\infty}^{\infty}|p'(x(u))|\mathrm{d}u\le\int_{-\infty}^{\infty}|-2ue^{-u^2}|\mathrm{d}u\le 2<\infty.$$

Hence, we get

$$\int_{-\infty}^{\infty}\big[|h'(x(u))|+|p'(x(u))|\big]\mathrm{d}u<\infty.$$

Note that

$$g(y)=36y+\frac{y}{1+|y|},\quad g(0)=0.$$

Then, we have

$$\frac{g(y)}{y}\ge 36,$$

and

$$g'(y)=36+\frac{1}{(1+|y|)^2}\le 36+\Big|\frac{1}{(1+|y|)^2}\Big|\le 37=L_2.$$

Also, the function

$$f(x)=\frac{3}{2}x+\frac{1}{2}\sin x,\quad f(0)=0,$$

and

$$\frac{f(x)}{x}=\frac{3}{2}+\frac{1}{2}\frac{\sin x}{x}\ge 1=b_3.$$

Therefore, we get

$$|f'(x)| = \left|\frac{3}{2} + \frac{1}{2}\cos x\right| \le 2 = L_1,$$

so

$$\sup\{f'(x)\} = 2 = b_1.$$

Hence, we have

$$\begin{aligned}\gamma &< \min\Big\{\frac{(1-\beta)(a_1b_2-a_2b_1)}{2\mu(L_1+L_2)(1-\beta)+L_1(1+a_2\mu)}, \frac{a_2(1-\beta)(a_1b_2-a_2b_1)}{2b_2\big\{L_2\{1+\mu a_2+a_2^2(1-\beta)\}+L_1a_2^2(1-\beta)\big\}}\Big\}\\ &= \min\Big\{\frac{21}{25}, 2.3\times 10^{-3}\Big\}.\end{aligned}$$

So all conditions of Theorem 2.3 hold, therefore the zero solution to (4.1) is uniformly asymptotically stable.

Example 4.2 In this example we shall study the boundedness of a third-order nonlinear DDE of the following form

$$\begin{aligned}&\Big[\Big(\frac{e^x}{1+e^{2x}}+\frac{3}{2}\Big)x''(t)\Big]' + \Big[\Big(\frac{2}{3}+e^{-x^2}\Big)x'(t)\Big]' + \Big[36x'(t-r(t)) + \frac{x'(t-r(t))}{1+|x'(t-r(t))|}\Big]\\ &+\frac{3}{2}x(t-r(t)) + \frac{1}{2}\sin(x(t-r(t))) = \frac{e^{-t}}{1+x^2+y^2+z^2}.\end{aligned} \tag{4.2}$$

Note that

$$e(t) = \frac{e^{-t}}{1+x^2+y^2+z^2},$$

then

$$|e(t)| \le e^{-t} = q(t), \quad \int_0^\infty q(t)dt = \int_0^\infty e^{-s}ds = 1 < \infty,$$

so $q(t) \in L^1(0,\infty)$. It follows that

$$\frac{dV}{dt} \le 2k_2q(t) + k_3[G(t)+q(t)]V, \quad k_3 = \max\{k_1, k_2D_1^{-1}\}, \tag{4.3}$$

where

$$\begin{aligned}\int_0^\infty G(t)dt &= \int_0^\infty [|\theta_1| + |\theta_2|]dt = \int_0^\infty [|h'(u)| + |p'(u)|]du\\ &= \int_0^\infty \Big[\Big|\frac{e^u - e^{3u}}{(1+e^{2u})^2}\Big| + \big|-2ue^{-u^2}\big|\Big]dt = \frac{1}{2}+1 = \frac{3}{2} < \infty.\end{aligned}$$

Integrating (4.3) from 0 to t, using the fact that $\int_0^\infty q(t)dt < \infty$, $\int_0^\infty G(t)dt < \infty$, we obtain

$$V(x_t,y_t,z_t) \le \Big[V(x_0,y_0,z_0) + 2k_2\int_0^\infty q(s)ds\Big]\exp\Big(k_3\int_0^\infty (G(t)+q(t))dt\Big) = c < \infty,$$

where $k_2 = \max\left\{\mu, \frac{1}{a_2}\right\} = \max\left\{\frac{1}{4}, \frac{2}{3}\right\} = \frac{2}{3}$. Then

$$V(x_t, y_t, z_t) \leq \Big[V(x_0, y_0, z_0) + \frac{2}{3}\Big] \exp\Big\{k_3\Big(\frac{3}{2} + 1\Big)\Big\} < \infty.$$

Hence, we can conclude that all solutions to equation (4.2) are bounded.

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