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Convergence of an Embedded Exponential-Type Low-Regularity Integrators for the KdV Equation without Loss of Regularity

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Abstract. In this paper, we study the convergence rate of an Embedded exponential-type low-regularity integrator (ELRI) for the Korteweg-de Vries equation. We develop some new harmonic analysis techniques to handle the "stability" issue. In particular, we use a new stability estimate which allows us to avoid the use of the fractional Leibniz inequality,

$$\left|\langle J^\gamma \partial_x(fg), J^\gamma f\rangle\right| \lesssim \|f\|_{H^\gamma}^2 \|g\|_{H^{\gamma+1}},$$

and replace it by suitable inequalities without loss of regularity. Based on these techniques, we prove that the ELRI scheme proposed in [41] provides $\frac{1}{2}$-order convergence accuracy in H^γ for any initial data belonging to H^γ with $\gamma > \frac{3}{2}$, which does not require any additional derivative assumptions.

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1 Introduction

The Korteweg-de Vries (KdV) equation arises as a model equation from the weakly nonlinear long waves and describes the propagation of shallow water waves in a channel [23]. It has taken a wide range of applications in a diverse field of the industries, especially in terms of application and technology. In this paper, we consider the KdV equation with periodic boundary conditions

$$\begin{cases} \partial_t u(t,x)+\partial_x^3 u(t,x)=\dfrac{1}{2}\partial_x(u(t,x))^2, & t>0, \quad x\in\mathbb{T}, \\ u(0,x)=u_0(x), & x\in\mathbb{T}, \end{cases} \tag{1.1}$$

where $\mathbb{T}=(0,2\pi)$, $u=u(t,x):\mathbb{R}^+\times\mathbb{T}\to\mathbb{R}$ is the unknown and $u_0\in H^{s_0}(\mathbb{T})$ with some $0\le s_0<\infty$ is a given initial data.

Many authors have studied the initial value problem of the KdV equation both on the real line and in the period case, and established the global well-posedness in H^s for $s\ge -1$; see [4, 18, 20]. The numerical solution of the KdV equation has been important in a wide range of fields. One interesting question in the numerical solution of the KdV equation is how much regularity is required in order to have certain desired convergence rates. Correspondingly, many numerical methods and numerical analysis were developed to address this question, including finite difference methods [5, 17, 21, 37], finite element methods [1, 6, 38], operator splitting [14–16, 36], spectral methods [7, 28, 29, 35], discontinuous Galerkin methods [26, 42] and exponential integrators [2, 11, 12].

Among the many numerical time integration methods for time-dependent partial differential equations (PDEs), the splitting methods are very popular in many classic studies. We refer the readers to [8, 13, 30] for an extensive overview of splitting methods. As far as we know, operator splitting methods for the KdV equation (often referred to as fractional-step methods) first appeared in [36] and were analysed rigorously in [15]. Operator splitting methods have been developed into a systematic approach for constructing time-stepping methods for evolutionary PDEs. In particular, Holden et al. [14, 16] proved that the Godunov and Strang splitting methods for the KdV equation converge with the first-order and the second-order rates in H^γ with $\gamma\ge 1$, if the initial datum belong to $H^{\gamma+3}$ and $H^{\gamma+5}$, respectively. For the nonlinear Schrödinger equation (NLS), Lubich [27] proved that for the initial data in H^4, the Strang splitting scheme provides the first-order and the second-order convergence in H^2 and L^2, respectively. In addition to the splitting method, exponential integrators is also a very effective numerical method for solving partial differential equations including hyperbolic and parabolic problems [9, 10]. In particular, Hochbruck and Ostermann [11] presented some typical applications that illustrate the computational benefits of exponential integrators.

These classical numerical methods greatly promote the development of numerical computation. However, it needs a large enough regularization requirement to achieve optimal convergence. Thus more and more attention has been paid recently to so-called low-regularity integrators (LRIs) that are based on the exponential integrators. This novel method has been used to obtain numerical solutions of many kinds of equations and obtained relatively ideal results. For the cubic nonlinear Schrödinger equation, the first-order convergence rate in H^γ have been achieved under $H^{\gamma+1}$-data by Ostermann and Schratz [33]. The second-order convergence rate in H^γ is proved under $H^{\gamma+2}$-data in one dimension and $H^{\gamma+3}$-data in high dimensions [22], respectively. More recently, Wu and Yao [39] proposed a new scheme which provides the first-order accuracy without loss of regularity in one dimensional case, that is, the first-order convergence in $H^\gamma(\mathbb{T})$ for $H^\gamma(\mathbb{T})$-data. Moreover, the algorithm in [39] almost preserves the mass of the numerical solution. Further, Li and Wu [25] constructed a fully discrete low-regularity integrator which has first-order convergence (up to a logarithmic factor) in $L^2(\mathbb{T})$ in both time and space for $H^1(\mathbb{T})$ initial data. For the KdV equation, Hofmanová and Schratz [12] proposed an exponential-type integrator and proved the first-order convergence in H^1 for initial data in H^3. Then based on a classical Lawson-type exponential integrator, Ostermann and Su [34] proposed a Fourier pseudospectral method to prove first-order convergence in both space and time under a mild Courant-Friedrichs-Lewy condition. Based on the scheme that was proposed in [12], Wu and Zhao [40] obtained the second-order convergence result in H^γ for initial data in $H^{\gamma+4}$ for the KdV equation.

Very recently, Wu and Zhao [41] further improved these results and established first-order and second-order convergence in H^γ under $H^{\gamma+1}$-data and $H^{\gamma+3}$-data respectively by introducing the Embedded exponential-type low-regularity integrators (ELRIs). That is, for any $\gamma>\frac{1}{2}$,

$$\|u(t_n,\cdot)-u^n\|_{H^\gamma}\le C\tau, \qquad n=0,1,\cdots,\frac{T}{\tau}, \tag{1.2}$$

where τ denotes the time step, u^n denotes the numerical solution at $t_n=n\tau$ and the constants τ_0 and C depend only on T and $\|u\|_{L^\infty((0,T);H^{\gamma+1})}$.

In [41], the first-order ELRI is stated as

$$u^n(x)=\tilde{u}^n(x+\widehat{u_0}(0)t_n)+\widehat{u_0}(0), \quad \tilde{u}^{n+1}=\Psi(\tilde{u}^n) \qquad \text{for } \ n=0,1,\cdots,\frac{T}{\tau}-1, \tag{1.3}$$

with $u^0=u_0$, where

$$\Psi(f)=\mathrm{e}^{-\tau\partial_x^3}f-\frac{1}{6}\mathrm{e}^{-\tau\partial_x^3}\left(\partial_x^{-1}f\right)^2+\frac{1}{6}\left(\mathrm{e}^{-\tau\partial_x^3}\partial_x^{-1}f\right)^2$$

$$
\begin{aligned}
&+\frac{1}{18}\mathbb{P}\Big[\mathrm{e}^{-\tau\partial_x^3}\partial_x^{-1}f\cdot\partial_x^{-1}\Big(\mathrm{e}^{-\tau\partial_x^3}\partial_x^{-1}f\Big)^2\Big]-\frac{1}{18}\mathbb{P}\Big[\mathrm{e}^{-\tau\partial_x^3}\partial_x^{-1}f\cdot\mathrm{e}^{-\tau\partial_x^3}\partial_x^{-1}\big(\partial_x^{-1}f\big)^2\Big]\\
&+\frac{1}{54}\mathrm{e}^{-\tau\partial_x^3}\partial_x^{-1}\big(\partial_x^{-1}f\big)^3-\frac{1}{54}\partial_x^{-1}\Big(\mathrm{e}^{-\tau\partial_x^3}\partial_x^{-1}f\Big)^3\\
&+\frac{\tau}{12\pi}\mathrm{e}^{-\tau\partial_x^3}\partial_x^{-1}f\int_{\mathbb{T}}\big(f\big)^2dx-\frac{\tau}{18}\partial_x^{-1}\mathrm{e}^{-\tau\partial_x^3}\big(f\big)^3.
\end{aligned}
$$

Here $\mathbb{P}$ is the orthogonal projection onto mean zero functions

$$\mathbb{P}f(x)=f(x)-\frac{1}{2\pi}\int_{\mathbb{T}}f\,dx.$$

The purpose of this investigation is to obtain the fractional order convergence without any derivative loss based on the same low-regularity integrator. More precisely, we are aiming to show that for any $\gamma>\frac{3}{2}$,

$$\|u(t_n,\cdot)-u^n\|_{H^\gamma}\le C\tau^{\frac{1}{2}},\qquad n=0,1,\cdots,\frac{T}{\tau},$$

where the constants τ_0 and C depend only on T and $\|u\|_{L^\infty((0,T);H^\gamma)}$.

The KdV equation, because of the derivative in the nonlinearity, is much harder than the nonlinear Schrödinger equation to obtain the convergence of the scheme without loss of the regularity. Now we explain our argument briefly. The main ingredients include the following two aspects. First, we establish that for any $\gamma>\frac{1}{2}$,

$$\|u(t_{n+1})-\Psi(u(t_n))\|_{H^\gamma}\le C\Big(\|u(t_n)\|_{H^\gamma}\Big)\tau^{\frac{3}{2}}.\tag{1.4}$$

This is regarded as the local error estimate. It is derived from some bilinear estimates based on the ingenious harmonic analysis, see Lemma 2.3 below.

Second, the handling of "stability" issues. As in the standard way, one shall prove the stability estimate:

$$\|\Psi(u^n)-\Psi(u(t_n))\|_{H^\gamma}\le\Big(1+\tau C\big(\|u(t_n)\|_{H^\gamma}\big)\Big)\|u^n-u(t_n)\|_{H^\gamma}.\tag{1.5}$$

Then the following fractional Leibniz inequality is needed:

$$\big|\big\langle J^\gamma\partial_x\big(fg\big),J^\gamma f\big\rangle\big|\lesssim\big\|f\big\|_{H^\gamma}^2\big\|g\big\|_{H^{\gamma+1}}.\tag{1.6}$$

Here $J^s=(1-\partial_{xx})^{\frac{s}{2}}$. However, in our setting, we only have the information that $f,g\in H^\gamma$. It is worth noting that the loss of the regularity is essential in the inequality (1.6). More precisely, the left-hand side in (1.6) can not be controlled by $\|f\|_{H^\gamma}$ and

$\|g\|_{H^\gamma}$ for any $\gamma\in\mathbb{R}$. Hence, the inequality (1.6) is of no use to us. This creates a fundamental difficulty for the target without the derivative loss.

In order to avoid using the inequality (1.6), we shall prove the following stability estimate instead:

$$\|\Psi(u^n)-\Psi(u(t_n))\|_{H^\gamma}^2\leq(1+C\tau)\|u^n-u(t_n)\|_{H^\gamma}^2+C\tau^{\frac{3}{2}}\|u^n-u(t_n)\|_{H^\gamma}. \tag{1.7}$$

Compared with the standard stability estimate (1.5), the estimate (1.7) does not require the square of the error $\|u^n-u(t_n)\|_{H^\gamma}$.

To prove (1.7), it reduces to show another type of the fractional Leibniz inequality involving on the linear flow:

$$\begin{aligned}&\Big|\int_0^\tau\big\langle J^\gamma e^{s\partial_x^3}\big(e^{-s\partial_x^3}f\cdot e^{-s\partial_x^3}\partial_x g\big),J^\gamma f\big\rangle ds\Big|\\ \lesssim&\tau^{\frac{1}{2}}\big\|g\big\|_{H^\gamma}\big\|f\big\|_{H^{0+}}\big\|f\big\|_{H^\gamma}+\tau\big\|g\big\|_{H^\gamma}\big\|f\big\|_{H^\gamma}^2.\end{aligned}\tag{1.8}$$

Thanks to the presence of the linear flow $e^{\pm s\partial_x^3}$, the inequality can be obtained from the smooth effect.

Applying (1.8), the stable part can be controlled by

$$\tau^{\frac{1}{2}}\big\|u^n-u(t_n)\big\|_{H^{0+}}\big\|u^n-u(t_n)\big\|_{H^\gamma}$$

with other easy treated terms. Then we use the convergence result (1.2) to establish the desired estimate (1.7).

Now, we state the convergence theorem of the presented (semi-discretized) ELRI method given in (1.3).

Theorem 1.1. *Let u^n be the numerical solution* (1.3) *of Eq.* (1.1) *up to some fixed time $T>0$. Under assumption that $u_0\in H^\gamma(\mathbb{T})$ for some $\gamma>\frac{3}{2}$, there exist constants $\tau_0,C>0$ such that for any $0<\tau\leq\tau_0$,*

$$\|u(t_n,\cdot)-u^n\|_{H^\gamma}\leq C\tau^{\frac{1}{2}},\qquad n=0,1,\cdots,\frac{T}{\tau}, \tag{1.9}$$

where the constants τ_0 and C depend only on T and $\|u\|_{L^\infty((0,T);H^\gamma)}$.

This research study fills a gap in the literature that optimal fractional order convergence without any derivative loss.

The paper is organized as follows. In Section 2, we give some notations and some useful lemmas. In Section 3, we present the local error and stability estimates and prove Theorem 1.1.

2 Preliminary

2.1 Some notations

We use $A\lesssim B$ or $B\gtrsim A$ to denote the statement that $A\leq CB$ for some absolute constant $C>0$ which may vary from line to line but is independent of τ or n, and we denote $A\sim B$ for $A\lesssim B\lesssim A$. We define $(d\xi)$ to be the normalized counting measure on $\mathbb{Z}$ such that

$$\int_{\mathbb{Z}} a(\xi)\,(d\xi)=\sum_{\xi\in\mathbb{Z}} a(\xi).$$

Sometimes, the subscript $\mathbb{Z}$ is omitted. The Fourier transform of a function f on $\mathbb{T}$ is defined by

$$\hat{f}(\xi)=\frac{1}{2\pi}\int_{\mathbb{T}} e^{-ix\xi}f(x)\,dx,$$

and thus the Fourier inversion formula is

$$f(x)=\int_{\mathbb{Z}} e^{ix\xi}\hat{f}(\xi)\,(d\xi).$$

Then the following usual properties of the Fourier transform hold:

$$\begin{aligned}
&\|f\|_{L^2(\mathbb{T})}=\sqrt{2\pi}\big\|\hat{f}\big\|_{L^2((d\xi))} && \text{(Plancherel)},\\
&\langle f,g\rangle=\int_{\mathbb{T}} f(x)\overline{g(x)}\,dx=2\pi\int_{\mathbb{Z}}\hat{f}(\xi)\overline{\hat{g}(\xi)}\,(d\xi) && \text{(Parseval)},\\
&\widehat{(fg)}(\xi)=\int_{\mathbb{Z}}\hat{f}(\xi-\xi_1)\hat{g}(\xi_1)\,(d\xi_1) && \text{(Convolution)}.
\end{aligned}$$

The Sobolev space $H^s(\mathbb{T})$ for $s\geq 0$ has the equivalent norm,

$$\big\|f\big\|_{H^s(\mathbb{T})}=\big\|J^s f\big\|_{L^2(\mathbb{T})}=\sqrt{2\pi}\Big\|(1+\xi^2)^{\frac{s}{2}}\hat{f}(\xi)\Big\|_{L^2((d\xi))},$$

where we denote the operator

$$J^s=(1-\partial_{xx})^{\frac{s}{2}}.$$

Moreover, we denote ∂_x^{-1} to be the operator defined by

$$\widehat{\partial_x^{-1}f}(\xi)=\begin{cases}(i\xi)^{-1}\hat{f}(\xi), & \text{when} \quad \xi\neq 0,\\ 0, & \text{when} \quad \xi=0.\end{cases} \tag{2.1}$$

For simplicity, we denote

$$\alpha_{k+1}=\xi^3-\xi_1^3-\cdots-\xi_k^3, \qquad k\geq 0.$$

Then if $\xi=\xi_1+\cdots+\xi_k$, we have that

$$\alpha_3=3\xi\xi_1\xi_2, \tag{2.2a}$$

$$\alpha_4=3(\xi\xi_1\xi_2+\xi\xi_1\xi_3+\xi\xi_2\xi_3-\xi_1\xi_2\xi_3). \tag{2.2b}$$

For convenience, in the following, we shall assume the zero-mode/average of the initial value of (1.1) is zero, that is, $\hat{u}_0(0)=0$. Otherwise, we may consider to replace u with

$$\tilde{u}(t,x):=u(t,x-\hat{u}_0(0)t)-\hat{u}_0(0)$$

and one may note that $\tilde{u}$ also obeys the same KdV equation of (1.1) with initial data $\tilde{u}_0:=u_0-\hat{u}_0(0)$. Then we can define the modified (with the new initial value) approximation v^n to the original solution $v(t_n)$ with the same exponential-type integration scheme. Furthermore, by the conservation law: $\int_{\mathbb{T}}u(t,x)dx=\int_{\mathbb{T}}u(0,x)dx$, we have that $\hat{u}(t,0)=0$ for any $t\geq 0$. Accordingly, the numerical approximation u^n defined above, preserves the mass, i.e., $\widehat{u^n}(0)=0$ for any $n=1,2,\cdots,T/\tau$.

2.2 Some preliminary estimates

First, we will frequently apply the following Kato-Ponce inequality (simple version), which was originally proved in [19] and an important progress in the endpoint case was made in [3, 24] very recently.

Lemma 2.1 (Kato-Ponce inequality)**.** *The following inequalities hold:*

(i) For any $\gamma>\frac{1}{2}$, $f,g\in H^\gamma$, then

$$\|J^\gamma(fg)\|_{L^2}\lesssim\|f\|_{H^\gamma}\|g\|_{H^\gamma}.$$

(ii) For any $\gamma\geq 0$, $\gamma_1>\frac{1}{2}$, $f\in H^{\gamma+\gamma_1}$, $g\in H^\gamma$, then

$$\|J^\gamma(fg)\|_{L^2}\lesssim\|f\|_{H^{\gamma+\gamma_1}}\|g\|_{H^\gamma}.$$

Based on the above inequalities, we can derive two lemmas as follows, which have been proved in [40, 41].

Lemma 2.2. *The following inequalities hold:*

(i) *For any* $\gamma>\frac{3}{2}$, $f\in H^\gamma$, $g\in H^\gamma$ *then*

$$\big|\big\langle J^\gamma\big(\partial_x f\cdot g\big),J^\gamma f\big\rangle\big|\lesssim\big\|f\big\|_{H^\gamma}^2\big\|g\big\|_{H^\gamma}.$$

(ii) *For any* $\gamma>\frac{3}{2}$, $f\in H^\gamma$, *then*

$$\big\langle J^\gamma\partial_x\big(f^2\big),J^\gamma f\big\rangle\lesssim\|f\|_{H^\gamma}^3.$$

Lemma 2.3. *The following inequalities hold:*

(i) *Let the space-time functions* $f,g\in L_t^\infty H_x^\gamma$ *and* $\partial_t f,\partial_t g\in L_t^\infty H_x^{\gamma-1}$ *for* $\gamma>\frac{1}{2}$ *with* $\hat f(0)=0$, $\hat g(0)=0$, *then*

$$\begin{aligned}&\Big\|\int_0^\tau \mathrm{e}^{t\partial_x^3}\partial_x\Big(\mathrm{e}^{-t\partial_x^3}f(t)\cdot\mathrm{e}^{-t\partial_x^3}g(t)\Big)\,dt\Big\|_{H^\gamma}\\&\lesssim\sqrt{\tau}\|f\|_{L_t^\infty H_x^\gamma}\|g\|_{L_t^\infty H_x^\gamma}+\tau\Big(\big\|\partial_t f\big\|_{L_t^\infty H_x^{\gamma-1}}\big\|g\big\|_{L_t^\infty H_x^{\gamma-1}}\\&\qquad+\big\|f\big\|_{L_t^\infty H_x^{\gamma-1}}\big\|\partial_t g\big\|_{L_t^\infty H_x^{\gamma-1}}\Big).\end{aligned}\tag{2.3}$$

(ii) *Let the space-time functions* $f,g\in L_t^\infty H_x^{\gamma-1}$ *and* $\partial_t f,\partial_t g\in L_t^\infty H_x^{\gamma-1}$ *for* $\gamma>\frac{1}{2}$ *with* $\hat f(0)=0$, $\hat g(0)=0$, *then*

$$\begin{aligned}&\Big\|\int_0^\tau \mathrm{e}^{t\partial_x^3}\partial_x\Big(\mathrm{e}^{-t\partial_x^3}f(t)\cdot\mathrm{e}^{-t\partial_x^3}g(t)\Big)\,dt\Big\|_{H^\gamma}\\&\lesssim\|f\|_{L_t^\infty H_x^{\gamma-1}}\|g\|_{L_t^\infty H_x^{\gamma-1}}+\tau\Big(\big\|\partial_t f\big\|_{L_t^\infty H_x^{\gamma-1}}\big\|g\big\|_{L_t^\infty H_x^{\gamma-1}}\\&\qquad+\big\|f\big\|_{L_t^\infty H_x^{\gamma-1}}\big\|\partial_t g\big\|_{L_t^\infty H_x^{\gamma-1}}\Big).\end{aligned}\tag{2.4}$$

Next we present a lemma that plays an important role in the stability estimate.

Lemma 2.4. *For any* $\gamma>\frac{3}{2}$, $\varepsilon>0$, *let the space functions* $f,g\in H_x^\gamma$ *with* $\hat f(0)=0$, $\hat g(0)=0$, *then*

$$\begin{aligned}&\Big|\int_0^\tau\big\langle J^\gamma \mathrm{e}^{s\partial_x^3}\big(\mathrm{e}^{-s\partial_x^3}f\cdot\mathrm{e}^{-s\partial_x^3}\partial_x g\big),J^\gamma f\big\rangle\,ds\Big|\\&\lesssim\tau^{\frac{1}{2}}\big\|g\big\|_{H^\gamma}\big\|f\big\|_{H^\varepsilon}\big\|f\big\|_{H^\gamma}+\tau\big\|g\big\|_{H^\gamma}\big\|f\big\|_{H^\gamma}^2.\end{aligned}$$

Proof. By Plancherel's identity, we get derictly

$$\begin{aligned}&\int_0^\tau \big\langle J^\gamma e^{s\partial_x^3}\big(e^{-s\partial_x^3}f\cdot e^{-s\partial_x^3}\partial_x g\big), J^\gamma f\big\rangle ds\\ =&2\pi\int_0^\tau\int_{\mathbb{Z}}\int_{\xi=\xi_1+\xi_2}\langle\xi\rangle^\gamma(i\xi_2)e^{-is\alpha_3}\hat{f}(\xi_1)\hat{g}(\xi_2)(d\xi_1)\langle\xi\rangle^\gamma\bar{\hat{f}}(\xi)(d\xi)ds.\end{aligned} \tag{2.5}$$

We assume that $\hat{f}$ and $\hat{g}$ are positive, otherwise one may replace them by $|\hat{f}|$ and $|\hat{g}|$. Now we consider two cases respectively.

Case 1: $|\xi|\le 2|\xi_1|$.

$$(2.5)\lesssim 2\pi\int_0^\tau\int_{\mathbb{Z}}\int_{\xi=\xi_1+\xi_2}\langle\xi_1\rangle^\gamma|\xi_2|\hat{f}(\xi_1)\hat{g}(\xi_2)(d\xi_1)\langle\xi\rangle^\gamma\bar{\hat{f}}(\xi)(d\xi)ds.$$

Note that the right term of the above inequality is equal to

$$\int_0^\tau \big\langle J^\gamma f\cdot|\nabla|g, J^\gamma f\big\rangle ds.$$

Hence by the Hölder inequality we get for any $\gamma_1>\frac{1}{2}$,

$$\Big|\int_0^\tau \big\langle J^\gamma e^{s\partial_x^3}\partial_x\big(e^{-s\partial_x^3}f\cdot e^{-s\partial_x^3}g\big), J^\gamma f\big\rangle ds\Big|\lesssim\tau\|g\|_{H^{\gamma_1+1}}\|f\|_{H^\gamma}^2.$$

Case 2: $|\xi|\ge 2|\xi_1|$. Integrating with respect to s for (2.5) and applying (2.2a), we get

$$(2.5)=2\pi\int_{\mathbb{Z}}\int_{\xi=\xi_1+\xi_2}\langle\xi\rangle^\gamma\big(e^{-i\tau\alpha_3}-1\big)\frac{i\xi_2}{-3i\xi\xi_1\xi_2}\hat{f}(\xi_1)\hat{g}(\xi_2)(d\xi_1)\langle\xi\rangle^\gamma\bar{\hat{f}}(\xi)(d\xi).$$

Note that

$$\big|e^{-i\tau\alpha_3}-1\big|\lesssim\tau^{\frac{1}{2}}|\alpha_3|^{\frac{1}{2}}\lesssim\tau^{\frac{1}{2}}|\xi|^{\frac{1}{2}}|\xi_1|^{\frac{1}{2}}|\xi_2|^{\frac{1}{2}}.$$

Therefore it gives that

$$|(2.5)|\lesssim\tau^{\frac{1}{2}}\int_{\mathbb{Z}}\int_{\xi=\xi_1+\xi_2}\langle\xi\rangle^\gamma\frac{|\xi_2|^{\frac{1}{2}}}{|\xi|^{\frac{1}{2}}|\xi_1|^{\frac{1}{2}}}\hat{f}(\xi_1)\hat{g}(\xi_2)(d\xi_1)\langle\xi\rangle^\gamma\bar{\hat{f}}(\xi)(d\xi).$$

Since $|\xi|\ge 2|\xi_1|$, we have that $|\xi|\sim|\xi_2|$. The right term of the above inequality is controlled by

$$\tau^{\frac{1}{2}}\int_{\mathbb{Z}}\int_{\xi=\xi_1+\xi_2}\langle\xi_2\rangle^\gamma\frac{1}{|\xi_1|^{\frac{1}{2}}}\hat{f}(\xi_1)\hat{g}(\xi_2)(d\xi_1)\langle\xi\rangle^\gamma\bar{\hat{f}}(\xi)(d\xi)=\frac{\tau^{\frac{1}{2}}}{2\pi}\big\langle|\nabla|^{-\frac{1}{2}}f\cdot J^\gamma g, J^\gamma f\big\rangle.$$

Hence by the Hölder and Sobolev inequalities, we derive that

$$\Big|\int_0^\tau \big\langle J^\gamma \mathrm{e}^{s\partial_x^3}\big(\mathrm{e}^{-s\partial_x^3}f\cdot \mathrm{e}^{-s\partial_x^3}\partial_x g\big), J^\gamma f\big\rangle ds\Big| \lesssim \tau^{\frac12}\big\|g\big\|_{H^\gamma}\big\|f\big\|_{H^\varepsilon}\big\|f\big\|_{H^\gamma}.$$

Combining these two cases, we finish the proof. $\square$

We define the operator $\mathcal{A}_n$ as

$$\mathcal{F}\big(\mathcal{A}_n(f_1,f_2,f_3)\big)(t,\xi)=\begin{cases} 0, & \text{if } \xi=0,\\ (i\xi)^{-1}\displaystyle\int_{\xi=\xi_1+\xi_2+\xi_3}\Big(\mathrm{e}^{-i(t_n+t)\alpha_4}-\mathrm{e}^{-it_n\alpha_4}\Big) & \\ \qquad\cdot\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\hat{f}_3(\xi_3)(d\xi_1)(d\xi_2), & \text{if } \xi\neq 0, \end{cases} \tag{2.6}$$

and for simplicity, we denote $\mathcal{A}_n(f)=\mathcal{A}_n(f,f,f)$ below.

Then we have the following estimate.

Lemma 2.5. *Let $\gamma_0>1$, and $f_1,f_2,f_3\in H^{\gamma_0}$, then for any $0\le t\le\tau$,*

$$\big\|\mathcal{A}_n(f_1,f_2,f_3)\big\|_{H^{\gamma_0}}\lesssim\tau^{\frac12}\big\|f_1\big\|_{H^{\gamma_0}}\big\|f_2\big\|_{H^{\gamma_0}}\big\|f_3\big\|_{H^{\gamma_0}}.$$

Proof. We assume that $\hat{f}_j$, $j=1,2,3$ are positive, otherwise one may replace them by $|\hat{f}_j|$. Note that

$$\Big|\mathrm{e}^{-i(t_n+t)\alpha_4}-\mathrm{e}^{-it_n\alpha_4}\Big|\le Ct^{\frac12}|\alpha_4|^{\frac12}.$$

Hence we have

$$\big|\mathcal{F}\big(\mathcal{A}_n(f_1,f_2,f_3)\big)(t,\xi)\big|\le Ct^{\frac12}|\xi|^{-1}\int_{\xi=\xi_1+\xi_2+\xi_3}|\alpha_4|^{\frac12}\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\hat{f}_3(\xi_3)(d\xi_1)(d\xi_2).$$

By (2.2b), we get

$$\begin{aligned}&|\xi|^{\gamma_0}\big|\mathcal{F}\big(\mathcal{A}_n(f_1,f_2,f_3)\big)(t,\xi)\big|\\ \lesssim& t^{\frac12}\int_{\xi=\xi_1+\xi_2+\xi_3}|\xi|^{\gamma_0-\frac12}\big(|\xi_1|^{\frac12}|\xi_2|^{\frac12}+|\xi_1|^{\frac12}|\xi_3|^{\frac12}+|\xi_2|^{\frac12}|\xi_3|^{\frac12}\big)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\hat{f}_3(\xi_3)(d\xi_1)(d\xi_2)\\ &+t^{\frac12}\int_{\xi=\xi_1+\xi_2+\xi_3}|\xi|^{\gamma_0-1}|\xi_1|^{\frac12}|\xi_2|^{\frac12}|\xi_3|^{\frac12}\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\hat{f}_3(\xi_3)(d\xi_1)(d\xi_2).\end{aligned}$$

By symmetry, without loss of generality, we assume that $|\xi_1|\ge|\xi_2|\ge|\xi_3|$, then

$$\begin{aligned}&|\xi|^{\gamma_0}\big|\mathcal{F}\big(\mathcal{A}_n(f_1,f_2,f_3)\big)(t,\xi)\big|\\ \lesssim& t^{\frac12}\int_{\xi=\xi_1+\xi_2+\xi_3,|\xi_1|\ge|\xi_2|\ge|\xi_3|}|\xi_1|^{\gamma_0}|\xi_2|^{\frac12}\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\hat{f}_3(\xi_3)(d\xi_1)(d\xi_2)\\ =&t^{\frac12}\mathcal{F}\Big(|\nabla|^{\gamma_0}f_1\cdot|\nabla|^{\frac12}f_2\cdot f_3\Big).\end{aligned}$$

Therefore, by Plancherel's identity and Lemma 2.1, we obtain that for any $\gamma_0 > 1$,

$$\big\|\mathcal{A}_n(f_1,f_2,f_3)\big\|_{H^{\gamma_0}} \lesssim \tau^{\frac{1}{2}} \big\|f_1\big\|_{H^{\gamma_0}} \big\|f_2\big\|_{H^{\gamma_0}} \big\|f_3\big\|_{H^{\gamma_0}}.$$

We get the desired result. □

3 Convergence result

3.1 The low-regularity integrator

By introducing the twisted variable $v := \mathrm{e}^{\partial_x^3 t} u$ and the Duhamel's formula at $t_n = n\tau$ with $\tau > 0$ the time step:

$$v(t_n+\tau,x) = v(t_n,x) + \frac{1}{2}\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3} \partial_x \Big(\mathrm{e}^{-(t_n+s)\partial_x^3} v(t_n+s,x)\Big)^2 ds. \tag{3.1}$$

As presented in [41], we rewrite v^{n+1} as

$$\begin{aligned} v^{n+1} = & v^n + \frac{1}{2}\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3} \partial_x \Big(\mathrm{e}^{-(t_n+s)\partial_x^3} v^n\Big)^2 ds \\ & + \frac{1}{2}\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3} \partial_x \Big(\mathrm{e}^{-(t_n+s)\partial_x^3} v^n \cdot \mathrm{e}^{-(t_n+s)\partial_x^3} F_n(v^n,s)\Big) ds \\ & + \frac{1}{18}\int_0^\tau \mathcal{A}_n(v^n)(s)\, ds, \end{aligned} \tag{3.2}$$

where F_n is defined as

$$F_n(v,s) = \int_0^s \mathrm{e}^{(t_n+t)\partial_x^3} \partial_x \Big(\mathrm{e}^{-(t_n+t)\partial_x^3} v\Big)^2 dt, \tag{3.3}$$

and $\mathcal{A}_n$ is defined in (2.6). Let

$$\begin{aligned} \Phi^n(v) = & v + \frac{1}{2}\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3} \partial_x \Big(\mathrm{e}^{-(t_n+s)\partial_x^3} v\Big)^2 ds \\ & + \frac{1}{2}\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3} \partial_x \Big(\mathrm{e}^{-(t_n+s)\partial_x^3} v \cdot \mathrm{e}^{-(t_n+s)\partial_x^3} F_n(v,s)\Big) ds + \frac{1}{18}\int_0^\tau \mathcal{A}_n(v)(s)\, ds. \end{aligned}$$

Then from (3.2), we have

$$v^{n+1} = \Phi^n(v^n).$$

Therefore we get

$$v^{n+1}-v(t_{n+1})=\Phi^n(v^n)-\Phi^n(v(t_n))+\Phi^n(v(t_n))-v(t_{n+1}).$$

Denote

$$\mathcal{L}^n=\Phi^n(v(t_n))-v(t_{n+1}),$$

then

$$v^{n+1}-v(t_{n+1})=\mathcal{L}^n+\Phi^n\big(v^n\big)-\Phi^n\big(v(t_n)\big).$$

3.2 Local error

The main result in this subsection is

Lemma 3.1. *Let $\gamma>\frac{1}{2}$ and $0\leq\tau\lesssim 1$, then*

$$\big\|\mathcal{L}^n\big\|_{H^\gamma}\leq C\tau^{\frac{3}{2}},$$

where the constant C depends only on $\|u\|_{L^\infty((0,T);H^\gamma)}$.

Proof. We split $\mathcal{L}^n$ into the following three parts as

$$\mathcal{L}^n=\mathcal{L}_1^n+\mathcal{L}_2^n+\mathcal{L}_3^n,$$

where

$$\begin{aligned}\mathcal{L}_1^n=&\frac{1}{2}\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x\Big(\mathrm{e}^{-(t_n+s)\partial_x^3}\Big(v(t_n)+\frac{1}{2}F_n\big(v(t_n),s\big)\Big)\Big)^2 ds\\&-\frac{1}{2}\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x\Big(\mathrm{e}^{-(t_n+s)\partial_x^3}v(t_n+s)\Big)^2 ds,\\ \mathcal{L}_2^n=&-\frac{1}{8}\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x\Big(\mathrm{e}^{-(t_n+s)\partial_x^3}F_n\big(v(t_n),s\big)\Big)^2 ds,\\ \mathcal{L}_3^n=&\frac{1}{18}\int_0^\tau \mathcal{A}_n\big(v(t_n)\big)(s)ds.\end{aligned}$$

For $\mathcal{L}_1^n$, we rewrite it as

$$\begin{aligned}\mathcal{L}_1^n=\frac{1}{2}\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x\bigg(&\mathrm{e}^{-(t_n+s)\partial_x^3}\Big(v(t_n)+\frac{1}{2}F_n\big(v(t_n),s\big)-v(t_n+s)\Big)\\&\cdot\mathrm{e}^{-(t_n+s)\partial_x^3}\Big(v(t_n)+\frac{1}{2}F_n\big(v(t_n),s\big)+v(t_n+s)\Big)\bigg)ds.\end{aligned}$$

Hence, by Lemma 2.3(ii), we have

$$\begin{aligned}\big\|\mathcal{L}_1^n\big\|_{H^\gamma} \lesssim & \Big\|v(t_n)+\frac{1}{2}F_n\big(v(t_n),s\big)+v(t_n+s)\Big\|_{L_t^\infty H_x^{\gamma-1}} \\ & \cdot\Big\|v(t_n)+\frac{1}{2}F_n\big(v(t_n),s\big)-v(t_n+s)\Big\|_{L_t^\infty H_x^{\gamma-1}} \\ & +\tau\Big\|v(t_n)+\frac{1}{2}F_n\big(v(t_n),s\big)+v(t_n+s)\Big\|_{L_t^\infty H_x^{\gamma-1}} \\ & \cdot\Big\|\partial_s\Big(v(t_n+s)-v(t_n)-\frac{1}{2}F_n\big(v(t_n),s\big)\Big)\Big\|_{L_t^\infty H_x^{\gamma-1}} \\ & +\tau\Big\|\partial_s\Big(v(t_n)+\frac{1}{2}F_n\big(v(t_n),s\big)+v(t_n+s)\Big)\Big\|_{L_t^\infty H_x^{\gamma-1}} \\ & \cdot\Big\|v(t_n+s)-v(t_n)-\frac{1}{2}F_n\big(v(t_n),s\big)\Big\|_{L_t^\infty H_x^{\gamma-1}}.\end{aligned}$$

From (3.1), we write

$$v(t_n+s)-v(t_n)=\frac{1}{2}\int_0^s \mathrm{e}^{(t_n+t)\partial_x^3}\partial_x\Big(\mathrm{e}^{-(t_n+t)\partial_x^3}v(t_n+t)\Big)^2dt. \tag{3.4}$$

Then using Lemma 2.3(i), we get

$$\big\|v(t_n+t)-v(t_n)\big\|_{L_t^\infty H_x^\gamma}\lesssim\sqrt{\tau}\big\|v(t)\big\|^2_{L_t^\infty H_x^\gamma}+2\tau\big\|\partial_t v(t)\big\|_{L_t^\infty H_x^{\gamma-1}}\big\|v(t)\big\|_{L_t^\infty H_x^{\gamma-1}}.$$

Note that

$$\partial_t v(t,x)=\frac{1}{2}\mathrm{e}^{t\partial_x^3}\partial_x\Big(\mathrm{e}^{-t\partial_x^3}v(t,x)\Big)^2,\qquad t\geq 0,\quad x\in\mathbb{T}.$$

Therefore for any $\gamma>\frac{1}{2}$,

$$\big\|\partial_t v(t)\big\|_{H^{\gamma-1}}\lesssim\big\|v(t)\big\|^2_{L_t^\infty H_x^\gamma}.$$

From the above estimates, we conclude that

$$\big\|v(t_n+t)-v(t_n)\big\|_{L_t^\infty H_x^\gamma}\lesssim\sqrt{\tau}\big\|v(t)\big\|^2_{L_t^\infty H_x^\gamma}+\tau\big\|v(t)\big\|^3_{L_t^\infty H_x^\gamma}. \tag{3.5}$$

From the definition of F_n in (3.3), the following equality holds:

$$\begin{aligned}&v(t_n+s)-v(t_n)-\frac{1}{2}F_n\big(v(t_n),s\big)\\ =&\frac{1}{2}\int_0^s \mathrm{e}^{(t_n+t)\partial_x^3}\partial_x\Big(\mathrm{e}^{-(t_n+t)\partial_x^3}\big(v(t_n+t)-v(t_n)\big)\cdot\mathrm{e}^{-(t_n+t)\partial_x^3}\big(v(t_n+t)+v(t_n)\big)\Big)dt.\end{aligned}$$

Hence, by Lemma 2.1, we have

$$\begin{aligned}&\Big\|v(t_n+s)-v(t_n)-\frac{1}{2}F_n\big(v(t_n),s\big)\Big\|_{H^{\gamma-1}}\\ \lesssim&\tau\big\|v(t_n+t)-v(t_n)\big\|_{L_t^\infty H_x^\gamma}\big\|v(t_n+t)+v(t_n)\big\|_{L_t^\infty H_x^\gamma}.\end{aligned}$$

Put (3.5) into the above inequality to get

$$\Big\|v(t_n+s)-v(t_n)-\frac{1}{2}F_n\big(v(t_n),s)\big)\Big\|_{H^{\gamma-1}}\lesssim\tau^{\frac{3}{2}}\big\|v(t)\big\|^3_{L_t^\infty H_x^\gamma}+\tau^2\big\|v(t)\big\|^4_{L_t^\infty H_x^\gamma}. \tag{3.6}$$

Moreover, from the definition of F_n in (3.3), we have that for any $0\le s\le\tau$,

$$\Big\|F_n\big(v(t_n),s\big)\Big\|_{H^{\gamma-1}}\lesssim\tau\big\|v(t)\big\|^2_{L_t^\infty H_x^\gamma},\qquad \Big\|\partial_sF_n\big(v(t_n),s\big)\Big\|_{H^{\gamma-1}}\lesssim\big\|v(t)\big\|^2_{L_t^\infty H_x^\gamma}. \tag{3.7}$$

Hence, using these estimates, we obtain

$$\big\|\mathcal{L}_1^n\big\|_{H^\gamma}\lesssim\tau^{\frac{3}{2}}. \tag{3.8}$$

For $\mathcal{L}_2^n$, by Lemma 2.3(ii), we obtain that

$$\|\mathcal{L}_2^n\|_{H^\gamma}\lesssim\|F_n\big(v(t_n),t\big)\|^2_{L_t^\infty H_x^{\gamma-1}}+\tau\|F_n\big(v(t_n),t\big)\|_{L_t^\infty H_x^{\gamma-1}}\big\|\partial_t\big(F_n\big(v(t_n),t\big)\big)\big\|_{L_t^\infty H_x^{\gamma-1}}.$$

Hence, by Lemma 2.1 and (3.7), we obtain that

$$\big\|\mathcal{L}_2^n\big\|_{H^\gamma}\lesssim\tau^2\big\|v(t)\big\|^4_{L_t^\infty H_x^\gamma}. \tag{3.9}$$

For $\mathcal{L}_n^3$, from Lemma 2.5, we get

$$\big\|\mathcal{L}_3^n\big\|_{H^\gamma}\lesssim\tau^{\frac{3}{2}}\big\|v(t)\big\|^3_{L_t^\infty H_x^\gamma}. \tag{3.10}$$

Together with (3.8)–(3.10), we prove the lemma. □

3.3 Stability

The main result in this subsection is

Lemma 3.2. *For any $\gamma>\frac{3}{2}$, then for any $n=0,\cdots,T/\tau-1$,*

$$\big\|\Phi^n(v^n)-\Phi^n(v(t_n))\big\|_{H^\gamma}\le(1+C\tau)\big\|v^n-v(t_n)\big\|_{H^\gamma}+C\tau\big\|v^n-v(t_n)\big\|^5_{H^\gamma}+C\tau^{\frac{3}{2}},$$

where the constant C depends only on $\|u\|_{L^\infty((0,T);H^\gamma)}$.

Proof. Note that

$$\Phi^n(v^n)-\Phi^n(v(t_n))=v^n-v(t_n)+\Phi_1^n+\Phi_2^n+\Phi_3^n,$$

where

$$\begin{aligned}
\Phi_1^n=&\frac{1}{2}\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x\left[\left(\mathrm{e}^{-(t_n+s)\partial_x^3}v^n\right)^2-\left(\mathrm{e}^{-(t_n+s)\partial_x^3}v(t_n)\right)^2\right]ds,\\
\Phi_2^n=&\frac{1}{2}\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x\Big(\mathrm{e}^{-(t_n+s)\partial_x^3}v^n\cdot \mathrm{e}^{-(t_n+s)\partial_x^3}F_n(v^n,s)\\
&\qquad -\mathrm{e}^{-(t_n+s)\partial_x^3}v(t_n)\cdot \mathrm{e}^{-(t_n+s)\partial_x^3}F_n(v(t_n),s)\Big)ds,\\
\Phi_3^n=&\frac{1}{18}\int_0^\tau \Big(\mathcal{A}_n\big(v^n\big)(s)-\mathcal{A}_n\big(v(t_n)\big)(s)\Big)ds.
\end{aligned}$$

For short, we denote $f_n=v^n-v(t_n)$, then

$$\begin{aligned}
&\big\|\Phi^n\big(v(t_n)\big)-\Phi^n\big(v^n\big)\big\|_{H^\gamma}^2\\
\le&\|f_n\|_{H^\gamma}^2+2\Big\langle J^\gamma\Phi_1^n,J^\gamma f_n\Big\rangle+2\|f_n\|_{H^\gamma}\big\|\Phi_2^n\big\|_{H^\gamma}+2\|f_n\|_{H^\gamma}\big\|\Phi_3^n\big\|_{H^\gamma}\\
&\quad+3\big\|\Phi_1^n\big\|_{H^\gamma}^2+3\big\|\Phi_2^n\big\|_{H^\gamma}^2+3\big\|\Phi_3^n\big\|_{H^\gamma}^2.
\end{aligned}$$

First, we rewrite Φ_1^n as

$$\begin{aligned}
\Phi_1^n=&\frac{1}{2}\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x\Big(\mathrm{e}^{-(t_n+s)\partial_x^3}f_n\Big)^2ds\\
&+\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x\Big(\mathrm{e}^{-(t_n+s)\partial_x^3}f_n\cdot \mathrm{e}^{-(t_n+s)\partial_x^3}v(t_n)\Big)ds.
\end{aligned}$$

Hence we get

$$\begin{aligned}
&\Big\langle J^\gamma\Phi_1^n,J^\gamma f_n\Big\rangle\\
=&\frac{1}{2}\int_0^\tau\Big\langle J^\gamma\partial_x\Big(\mathrm{e}^{-(t_n+s)\partial_x^3}f_n\Big)^2,J^\gamma \mathrm{e}^{-(t_n+s)\partial_x^3}f_n\Big\rangle ds \qquad (3.11)\\
&+\int_0^\tau\Big\langle J^\gamma\Big(\mathrm{e}^{-(t_n+s)\partial_x^3}\partial_x f_n\cdot \mathrm{e}^{-(t_n+s)\partial_x^3}v(t_n)\Big),J^\gamma \mathrm{e}^{-(t_n+s)\partial_x^3}f_n\Big\rangle ds \qquad (3.12)\\
&+\int_0^\tau\Big\langle J^\gamma\Big(\mathrm{e}^{-(t_n+s)\partial_x^3}f_n\cdot \mathrm{e}^{-(t_n+s)\partial_x^3}\partial_x v(t_n)\Big),J^\gamma \mathrm{e}^{-(t_n+s)\partial_x^3}f_n\Big\rangle ds. \qquad (3.13)
\end{aligned}$$

For (3.11), by Lemma 2.2(ii), we have

$$(3.11)\lesssim\tau\big\|f_n\big\|_{H^\gamma}^3.$$

For (3.12), by Lemma 2.2(i), we have

$$(3.12)\lesssim\tau\big\|f_n\big\|_{H^\gamma}^2\big\|v(t_n)\big\|_{H^\gamma}.$$

For (3.13), as the main difficult term, we use the key estimate Lemma 2.4 and choose $\varepsilon\in(0,\frac{1}{2})$ to deduce that for any $\gamma>\frac{3}{2}$

$$(3.13)\lesssim\tau^{\frac{1}{2}}\big\|v(t_n)\big\|_{H^\gamma}\big\|f_n\big\|_{H^\varepsilon}\big\|f_n\big\|_{H^\gamma}+\tau\big\|v(t_n)\big\|_{H^\gamma}\big\|f_n\big\|_{H^\gamma}^2. \tag{3.14}$$

From (1.2), we have that for any $\gamma>\frac{3}{2}$

$$\|u(t_n,\cdot)-u^n\|_{H^{\gamma-1}}\leq C\tau,\qquad n=0,1,\cdots,T/\tau,$$

where the constants τ_0 and C depend only on T and $\|u\|_{L^\infty((0,T);H^\gamma)}$.

Then put the above inequality into (3.14) to yield that

$$(3.13)\leq C\tau^{\frac{3}{2}}\big\|f_n\big\|_{H^\gamma}+C\tau\big\|f_n\big\|_{H^\gamma}^2.$$

Combining with the above estimates, we conclude that

$$\Big\langle J^\gamma\Phi_1^n,J^\gamma f_n\Big\rangle\leq C\Big(\tau\big\|f_n\big\|_{H^\gamma}^3+\tau\big\|f_n\big\|_{H^\gamma}^2+\tau^{\frac{3}{2}}\big\|f_n\big\|_{H^\gamma}\Big), \tag{3.15}$$

where the constant C depends only on $\|u\|_{L^\infty((0,T);H^\gamma)}$.

For $\big\|\Phi_1^n\big\|_{H^\gamma}$, from Lemma 2.3(i), we have

$$\big\|\Phi_1^n\big\|_{H^\gamma}\lesssim\sqrt{\tau}\Big(\big\|v(t_n)\big\|_{H^\gamma}\big\|f_n\big\|_{H^\gamma}+\big\|f_n\big\|_{H^\gamma}^2\Big). \tag{3.16}$$

For Φ_2^n, we rewrite it as

$$\begin{aligned}
&\Phi_2^n\\
=&\frac{1}{2}\int_0^\tau e^{(t_n+s)\partial_x^3}\partial_x\Big(e^{-(t_n+s)\partial_x^3}f_n\cdot e^{-(t_n+s)\partial_x^3}F_n(v(t_n),s)\Big)ds &(3.17)\\
&+\frac{1}{2}\int_0^\tau e^{(t_n+s)\partial_x^3}\partial_x\Big(e^{-(t_n+s)\partial_x^3}f_n\cdot e^{-(t_n+s)\partial_x^3}\big(F_n(v^n,s)-F_n(v(t_n),s)\big)\Big)ds &(3.18)\\
&+\frac{1}{2}\int_0^\tau e^{(t_n+s)\partial_x^3}\partial_x\Big(e^{-(t_n+s)\partial_x^3}v(t_n)\cdot e^{-(t_n+s)\partial_x^3}\big(F_n(v^n,s)-F_n(v(t_n),s)\big)\Big)ds. &(3.19)
\end{aligned}$$

For (3.17), using Lemma 2.3(ii), we have

$$\big\|(3.17)\big\|_{H^\gamma}\lesssim\big\|f_n\big\|_{H^{\gamma-1}}\big\|F_n(v(t_n),t)\big\|_{L_t^\infty H_x^{\gamma-1}}+\tau\big\|f_n\big\|_{H^{\gamma-1}}\big\|\partial_tF_n(v(t_n),t)\big\|_{L_t^\infty H_x^{\gamma-1}}.$$

This together with (3.7) yields that

$$\big\|(3.17)\big\|_{H^\gamma}\lesssim\tau\big\|v(t)\big\|^2_{L^\infty_t H^\gamma_x}\big\|f_n\big\|_{H^\gamma}. \tag{3.20}$$

For (3.18), using Lemma 2.3(ii), we have

$$\begin{aligned}\big\|(3.18)\big\|_{H^\gamma}\lesssim&\big\|f_n\big\|_{H^{\gamma-1}}\big\|F_n(v^n,t)-F_n(v(t_n),t)\big\|_{L^\infty_t H^{\gamma-1}_x}\\&+\tau\big\|f_n\big\|_{H^{\gamma-1}}\big\|\partial_t\big(F_n(v^n,t)-F_n(v(t_n),t)\big)\big\|_{L^\infty_t H^{\gamma-1}_x}.\end{aligned}$$

From (3.3) and Lemma 2.1,

$$\begin{aligned}&\big\|F_n(v^n,t)-F_n(v(t_n),t)\big\|_{L^\infty_t H^{\gamma-1}_x}\\\lesssim&\int_0^\tau\Big\|\big(\mathrm{e}^{-(t_n+t)\partial_x^3}v^n\big)^2-\big(\mathrm{e}^{-(t_n+t)\partial_x^3}v(t_n)\big)^2\Big\|_{H^\gamma}dt\\\lesssim&\tau\big\|f_n\big\|^2_{H^\gamma}+\tau\big\|v(t)\big\|_{L^\infty_t H^\gamma_x}\big\|f_n\big\|_{H^\gamma}.\end{aligned}$$

Similarly,

$$\big\|\partial_t\big(F_n(v^n,t)-F_n(v(t_n),t)\big)\big\|_{L^\infty_t H^{\gamma-1}_x}\lesssim\big\|f_n\big\|^2_{H^\gamma}+\big\|v(t)\big\|_{L^\infty_t H^\gamma_x}\big\|f_n\big\|_{H^\gamma}.$$

Using these two estimates, we get

$$\big\|(3.18)\big\|_{H^\gamma}\lesssim\tau\big\|f_n\big\|^3_{H^\gamma}+\tau\big\|v(t)\big\|_{L^\infty_t H^\gamma_x}\big\|f_n\big\|^2_{H^\gamma}. \tag{3.21}$$

Treating similarly as (3.18), we have

$$\big\|(3.19)\big\|_{H^\gamma}\lesssim\tau\big\|v(t)\big\|_{L^\infty_t H^\gamma_x}\big\|f_n\big\|^2_{H^\gamma}+\tau\big\|v(t)\big\|^2_{L^\infty_t H^\gamma_x}\big\|f_n\big\|_{H^\gamma}. \tag{3.22}$$

Together with the estimates in (3.20), (3.21) and (3.22), we get

$$\big\|\Phi_2^n\big\|_{H^\gamma}\le C\tau\Big(\big\|f_n\big\|_{H^\gamma}+\big\|f_n\big\|^3_{H^\gamma}\Big), \tag{3.23}$$

where the constant C depends only on $\|u\|_{L^\infty((0,T);H^\gamma)}$.

For $\big\|\Phi_3^n\big\|_{H^\gamma}$, from Lemma 2.5, we have

$$\big\|\Phi_3^n\big\|_{H^\gamma}\le C\tau^{\frac32}\Big(\big\|f_n\big\|_{H^\gamma}+\big\|f_n\big\|^3_{H^\gamma}\Big), \tag{3.24}$$

where the constant C depends only on $\|u\|_{L^\infty((0,T);H^\gamma)}$.

Combining (3.15), (3.16), (3.23) with (3.24), we obtain

$$\big\|\Phi^n\big(v(t_n)\big)-\Phi^n\big(v^n\big)\big\|_{H^\gamma}^2 \leq (1+C\tau)\|f_n\|_{H^\gamma}^2+C\tau\big\|f_n\big\|_{H^\gamma}^6+C\tau^{\frac{3}{2}}\big\|f_n\big\|_{H^\gamma}.$$

By the inequality

$$\sqrt{a+b}\leq\frac{b}{2\sqrt{a}}+\sqrt{a},$$

and choosing $a=(1+C\tau)\|f_n\|_{H^\gamma}^2$, $b=C\tau\big\|f_n\big\|_{H^\gamma}^6+C\tau^{\frac{3}{2}}\big\|f_n\big\|_{H^\gamma}$, we find the following inequality holds

$$\big\|\Phi^n(v^n)-\Phi^n(v(t_n))\big\|_{H^\gamma}\leq(1+C\tau)\big\|v^n-v(t_n)\big\|_{H^\gamma}+C\tau\big\|v^n-v(t_n)\big\|_{H^\gamma}^5+C\tau^{\frac{3}{2}}.$$

This gets the desired result. □

3.4 Proof of Theorem 1.1

Now, combining the local error estimate and the stability result, we give the proof of Theorem 1.1. From Lemma 3.1 and Lemma 3.2, there exits a constant $C>0$, such that for $0<\tau\leq 1$, we have

$$\begin{aligned}&\big\|v(t_{n+1})-v^{n+1}\big\|_{H^\gamma}\\ \leq& C\tau^{\frac{3}{2}}+(1+C\tau)\big\|v(t_n)-v^n\big\|_{H^\gamma}+C\tau\big\|v(t_n)-v^n\big\|_{H^\gamma}^5, \qquad n=0,1,\cdots,T/\tau-1,\end{aligned}$$

where C depends on T and $\|v\|_{L^\infty((0,T);H^\gamma)}$. By iteration and Gronwall's inequality, we get

$$\begin{aligned}&\big\|v(t_{n+1})-v^{n+1}\big\|_{H^\gamma}\\ \leq& C\tau^{\frac{3}{2}}\sum_{j=0}^{n}(1+C\tau)^j\leq C\tau^{\frac{1}{2}}, \qquad n=0,1,\cdots,T/\tau-1,\end{aligned}$$

which proves Theorem 1.1. □

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