

Boundary Homogenization of a Class of Obstacle Problems

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Abstract. We study the homogenization of a boundary obstacle problem on a $C^{1,\alpha}$ -domain D for some elliptic equations with uniformly elliptic coefficient matrices γ . For any $\epsilon \in \mathbb{R}_+$, $\partial D = \Gamma \cup \Sigma$, $\Gamma \cap \Sigma = \emptyset$ and $S_\epsilon \subset \Sigma$ with suitable assumptions, we prove that as ϵ tends to zero, the energy minimizer u^ϵ of $\int_D |\gamma \nabla u|^2 dx$, subject to $u \geq \varphi$ on S_ϵ , up to a subsequence, converges weakly in $H^1(D)$ to \tilde{u} , which minimizes the energy functional

$$\int_D |\gamma \nabla u|^2 + \int_\Sigma (u - \varphi)_-^2 \mu(x) dS_x,$$

where $\mu(x)$ depends on the structure of S_ϵ and φ is any given function in $C^\infty(\overline{D})$.

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1 Introduction

Let $D \subset \mathbb{R}^n$ ($n > 2$) be a bounded open subset, whose boundary satisfies

$$\partial D = \Gamma \cup \Sigma, \quad \Gamma \cap \Sigma = \emptyset \quad \text{and} \quad \partial D \in C^{1,\alpha}, \tag{1.1}$$

for some constant $\alpha \in (0, 1)$. For any $\epsilon \in \mathbb{R}_+$, let S_ϵ be a subset of Σ with some special structure, which will be specified later. Throughout, we assume:

- (a1) $\gamma(x) = (\gamma_{ij}(x))_{n \times n}$ is an $n \times n$ symmetric matrix-valued function on D and there exist two positive constants a and b with $a \leq b$ such that

$$aI \leq \gamma(x) \leq bI, \quad x \in D,$$

where I is the identity matrix;

- (a2) φ and ψ are both smooth functions defined on \overline{D} .

Consider the following variational problem

$$\inf_{v \in K} J(v), \tag{1.2}$$

where

$$J(v) := \int_D |\gamma \nabla v|^2 dx, \tag{1.3}$$

and

$$K := \{v \in H^1(D) : v|_\Gamma = \psi \quad \text{and} \quad v|_{S_\epsilon} \geq \varphi\}. \tag{1.4}$$

Let u^ϵ be the solution to the variational problem (1.1)–(1.3). Here we focus on the study of the asymptotic behavior of u^ϵ when $\epsilon \rightarrow 0$ under suitable assumptions on S_ϵ and Σ . That is, we are concerned with the boundary homogenization associated with the variational problem (1.2). This is an important problem with many practical applications. For instance, (1.2) can be used to describe the mathematical model for semipermeable membranes, where the function $\varphi(x)$ signifies an external pressure, and the set S_ϵ is considered as a subset of the boundary composed of the part through which the liquid passes on the semipermeable membrane. Interested readers may refer to [9] and the references therein for more details.

In case that $\gamma(x)$ is the identity matrix I , there is a long history in studying (1.2) with rich results in the literature. When the set S_ϵ lies inside D , the problem can be viewed as the homogenization of a variational problem on a perforated domain. For this problem, [6, 7] firstly considered the periodic homogenization and established that the limiting energy functional contains a strange term which depends on the

capacity of S_ϵ . Later [3] obtained the stochastic homogenization result under the setting of the stationary ergodic case. For more general energy structure, see [1] and [23] and the references cited therein. When $S_\epsilon \subset \partial D$, the problem becomes more tricky. If S_ϵ lies on the straight part of the boundary with some suitable assumptions, Caffarelli and Mellet [4] established the random homogenization result of (1.2) under the stationary ergodic setting. If S_ϵ and ∂D satisfy more general conditions, Yang [24] established the homogenization result, which contains an unusual term in the limiting functional.

In this paper, we considered the general case when γ is a variable matrix-valued function as prescribed in (a1), which is equivalent to introducing a Riemannian metric to the variational problem (1.1)–(1.3) in the domain D . Before stating the main result, we made the following assumptions on the precise structure S_ϵ in our setting.

First, for any $\epsilon > 0$, we define

$$T_\epsilon = \left(\bigcup_k B_{r_{\epsilon,k}}(x_{\epsilon,k}) \right) \cap D \quad \text{and} \quad S_\epsilon = \left(\bigcup_k B_{r_{\epsilon,k}}(x_{\epsilon,k}) \right) \cap \Sigma, \quad (1.5)$$

where $x_{\epsilon,k} \in \Sigma$. It is assumed that for any $k, \tilde{k} \in \mathbb{Z}^n$ and $(k \neq \tilde{k})$, there holds

$$|x_{\epsilon,k} - x_{\epsilon,\tilde{k}}| \geq 2\epsilon. \quad (1.6)$$

The assumption (1.6) implies that the centers of the balls are periodically distributed in the obstacle set Σ and those balls are mutually disjoint, which is a common and applicable situation in the field of homogenization theory. Note that the center points are taken on a compact set and there can be only a finite number of them and the number of balls is dependent on their capacity or volume.

Next we assume there exist two positive constants c_1 and c_2 (independent of k and ϵ) such that

$$r_{\epsilon,k} = \tilde{r}_{\epsilon,k} \epsilon^{\frac{n-1}{n-2}}, \quad (1.7)$$

where $0 < c_1 \leq \tilde{r}_{\epsilon,k} \leq c_2$. Clearly, the number of such balls is $\mathcal{O}(\epsilon^{1-n})$.

Then, we construct a class of density functions $\mu_\epsilon(x)$ on D as follows :

$$\mu_\epsilon(x) := \sum_k \frac{\left(\sum_{i=1}^{\infty} \tilde{C}_i \epsilon^{i-1} \right) \tilde{r}_{\epsilon,k}^{n-2}}{\epsilon \tilde{r}_{\epsilon,k}^{n-2} - 1} \nabla \cdot (\gamma^T \gamma x) \frac{1}{\epsilon} \chi_{B_\epsilon(x_{\epsilon,k})}(x), \quad (1.8)$$

where $\chi_{B_\epsilon(x_{\epsilon,k})}(x)$ is the characteristic function of $B_\epsilon(x_{\epsilon,k})$ and $\{\tilde{C}_i\}_{i=1}^{\infty}$ are bounded. We shall assume that μ_ϵ converges properly as $\epsilon \rightarrow 0$ as described in the following theorem.

The main result of this paper is given as follows.

Theorem 1.1. *Assume that as $\epsilon \rightarrow 0$, $\mu_\epsilon(x)dx \rightarrow \mu(x)dS_x$ in $H^{-1}(D)$, where $\mu(x)$ is some density function on Σ . If u^ϵ is the minimizer of (1.2), then as $\epsilon \rightarrow 0$, u^ϵ converges weakly to \tilde{u} in $H^1(D)$, where \tilde{u} minimizes*

$$J_\mu(v) = \int_D |\gamma \nabla v|^2 dx + c_n \int_\Sigma (v - \varphi)_-^2 \mu(x) dS_x$$

over

$$\tilde{K} = \{v \in H^1(D) : v|_\Gamma = \psi\}.$$

Here $(v - \varphi)_- = -\min\{v - \varphi, 0\}$ and c_n is a positive constant depending only on n .

Remark 1.1. We would like to point out that a similar assumption to the one in Theorem 1.1 was considered in [24]. However, the author in [24] considered the special case of Theorem 1.1 with $\gamma = I$. The presence of a general γ makes the corresponding analysis in proving Theorem 1.1 more challenging and subtle. Roughly speaking, we mainly construct the corrector $\omega_\epsilon(x)$ in Proposition 2.1 by means of the fundamental solution of the equation

$$\nabla \cdot (\gamma^T \gamma \nabla u) = 0.$$

Since this solution is associated with asymptotic expansion of $d_g(x, y)$ (see Section 3), we need to deal with multiple terms in the process of estimating the corrector ω_ϵ .

Finally, we would like to mention in passing a related inverse problem of reconstructing γ and the inaccessible part Σ by knowledge of v on Γ ; see [11–14, 19, 25] for related mathematical formulation as well as [2, 5, 8, 15–18, 21] for some background discussion. Our study paves the way for such an inverse problem study and we shall address it in our future work. The remaining part of this paper is arranged as follows. In Section 2, we introduce a key lemma for correctors, Proposition 2.1, with which Theorem 1.1 can be proved. In Section 3, we construct the corrector and show that it satisfies the conditions (2.1a)–(2.1d) in Proposition 2.1. In Section 4, we establish the limiting property of the corrector to finish the proof of Proposition 2.1.

In what follows, C shall signify a generic positive constant which may vary in different inequalities. The symbol “ \rightharpoonup ” denotes the notion of weak convergence.

2 Proof of the main theorem

The proof of the Theorem 1.1 critically relies on the following proposition, whose proof is postponed to Section 3–4.

Proposition 2.1. *For any $\epsilon > 0$, let T_ϵ and S_ϵ be defined by (1.5), and the density function $\mu_\epsilon(x)$ is given by (1.8). Assume that as $\epsilon \rightarrow 0$, $\mu_\epsilon(x)dx \rightarrow \mu(x)dS_x$ in $H^{-1}(D)$, where $\mu(x)$ is some density function on Σ . Then there exists a function $\omega_\epsilon(x) \in H^1(D)$ satisfying the following conditions:*

$$\omega_\epsilon(x) = 1, \quad \forall x \in T_\epsilon, \tag{2.1a}$$

$$\nabla \cdot (\gamma^T \gamma \nabla \omega_\epsilon) = 0, \quad \forall x \in D, \tag{2.1b}$$

$$\|\omega_\epsilon\|_{L^\infty(D)} \leq C, \tag{2.1c}$$

$$\omega_\epsilon(x) \rightarrow 0 \quad \text{in } H^1(D) \quad \text{as } \epsilon \rightarrow 0. \tag{2.1d}$$

Furthermore, given any function $v_\epsilon \in H^1(D)$ satisfying:

$$\begin{aligned} v_\epsilon(x) &\geq 0 && \text{for } x \in T_\epsilon, \quad \|v_\epsilon\|_{L^\infty} \leq C, \\ v_\epsilon &\rightharpoonup v && \text{in } H^1(D) \quad \text{as } \epsilon \rightarrow 0, \end{aligned}$$

then there holds:

$$\lim_{\epsilon \rightarrow 0} \int_D (\gamma \nabla \omega_\epsilon) \cdot (\gamma \nabla v_\epsilon) \phi dx \geq - \int_\Sigma v \phi \mu(x) dS_x \tag{2.2}$$

for any $\phi(x) \in C^\infty(\bar{D})$ with $\phi|_\Gamma = 0$. The equality in (2.2) also holds if $v_\epsilon = 0$ on T_ϵ .

From the above proposition, we have the following result :

Lemma 2.1. *For any $\phi \in \{\phi \in C^\infty(\bar{D}) : \phi|_\Gamma = 0\}$, we have that*

$$\lim_{\epsilon \rightarrow 0} \int_D |\gamma \nabla \omega_\epsilon|^2 \phi dx = \int_\Sigma \phi \mu(x) dS_x.$$

Proof. Set $v_\epsilon = 1 - \omega_\epsilon$, then we see obviously that $v_\epsilon|_{T_\epsilon} = 0$ and $v_\epsilon \rightarrow 1$ in $H^1(D)$. Using Proposition 2.1, one can readily show that

$$\lim_{\epsilon \rightarrow 0} \int_D |\gamma \nabla \omega_\epsilon|^2 \phi dx = \int_\Sigma \phi \mu(x) dS_x.$$

Thus, we complete the proof. □

By Lemma 2.1 and Proposition 2.1, we have the following property :

Lemma 2.2. *Assume that $u^\epsilon \rightharpoonup \tilde{u}$ in $H^1(D)$, then we have the following lower semi-continuous property :*

$$\liminf_{\epsilon \rightarrow 0} \int_D |\gamma \nabla u^\epsilon|^2 dx \geq \int_D |\gamma \nabla \tilde{u}|^2 dx + \int_\Sigma (\tilde{u} - \varphi)_-^2 \mu(x) dS_x.$$

Proof. Let us decompose $u^\epsilon = u_+^\epsilon - u_-^\epsilon$. Then we obviously have $u_+^\epsilon \rightharpoonup \tilde{u}_+$ in $H^1(D)$ ($u_-^\epsilon \rightharpoonup \tilde{u}_-$ in $H^1(D)$ respectively).

For u_+^ϵ , we apply the classical lower semi-continuity property :

$$\liminf_{\epsilon \rightarrow 0} \int_D |\gamma \nabla u_+^\epsilon|^2 dx \geq \int_D |\gamma \nabla \tilde{u}_+|^2 dx.$$

In order to prove Lemma 2.2, we need to prove the following modified lower semi-continuity property :

$$\liminf_{\epsilon \rightarrow 0} \int_D |\gamma \nabla u_-^\epsilon|^2 dx \geq \int_D |\gamma \nabla \tilde{u}_-|^2 dx + \int_\Sigma (\tilde{u} - \varphi)_-^2 \mu(x) dS_x.$$

Here we consider the following two estimates :

$$\liminf_{\epsilon \rightarrow 0} \frac{1}{2} \int_{\omega_\epsilon \leq \theta} |\gamma \nabla u_-^\epsilon|^2 dx \geq \int_D (\gamma \nabla \phi) \cdot (\gamma \nabla \tilde{u}_-) dx - \frac{1}{2} \int_D |\gamma \nabla \phi|^2 dx, \tag{2.3a}$$

$$\begin{aligned} \frac{1}{2} \int_{\omega_\epsilon > \theta} |\gamma \nabla u_-^\epsilon|^2 dx \geq & -\frac{1}{2} \int_D |\gamma \nabla \omega_\epsilon|^2 \tilde{\phi}^2 dx - \int_D (\gamma \nabla \omega_\epsilon) \cdot (\gamma \nabla u_-^\epsilon) \tilde{\phi} dx \\ & + C\theta + C\theta^{\frac{1}{2}}, \end{aligned} \tag{2.3b}$$

where θ is a positive (small) constant, and $\phi, \tilde{\phi}$ are two test functions (to be determined).

Suppose the above two estimates hold. Then we can use Proposition 2.1 and Lemma 2.1 to obtain

$$\begin{aligned} \liminf_{\epsilon \rightarrow 0} \int_D \frac{1}{2} |\gamma \nabla u^\epsilon|^2 dx \geq & \liminf_{\epsilon \rightarrow 0} \left(\frac{1}{2} \int_{\omega_\epsilon \leq \theta} |\gamma \nabla u_-^\epsilon|^2 dx + \frac{1}{2} \int_{\omega_\epsilon > \theta} |\gamma \nabla u_-^\epsilon|^2 dx \right) \\ \geq & \int_D (\gamma \nabla \phi) \cdot (\gamma \nabla \tilde{u}_-) dx - \frac{1}{2} \int_D |\gamma \nabla \phi|^2 dx \\ & - \frac{1}{2} \int_\Sigma \tilde{\phi}^2 \mu(x) dS_x + \int_\Sigma (\tilde{u} - \varphi)_- \tilde{\phi} \mu(x) dS_x + C\theta + C\theta^{\frac{1}{2}}. \end{aligned}$$

Since θ is arbitrarily small, we can further obtain

$$\begin{aligned} \liminf_{\epsilon \rightarrow 0} \frac{1}{2} \int_D |\gamma \nabla u^\epsilon|^2 dx \geq & \liminf_{\epsilon \rightarrow 0} \left(\frac{1}{2} \int_{\omega_\epsilon \leq \theta} |\gamma \nabla u_-^\epsilon|^2 dx + \frac{1}{2} \int_{\omega_\epsilon > \theta} |\gamma \nabla u_-^\epsilon|^2 dx \right) \\ \geq & \int_D (\gamma \nabla \phi) \cdot (\gamma \nabla \tilde{u}_-) dx - \frac{1}{2} \int_D |\gamma \nabla \phi|^2 dx \\ & - \frac{1}{2} \int_\Sigma \tilde{\phi}^2 \mu(x) dS_x + \int_\Sigma (\tilde{u} - \varphi)_- \tilde{\phi} \mu(x) dS_x. \end{aligned} \tag{2.4}$$

If we choose $\phi = \tilde{u}_-$ and $\tilde{\phi} = (\tilde{u} - \varphi)_-$ in (2.4), we can readily have

$$\liminf_{\epsilon \rightarrow 0} \int_D |\gamma \nabla u^\epsilon|^2 dx \geq \int_D |\gamma \nabla \tilde{u}|^2 dx + \int_\Sigma (\tilde{u} - \varphi)_-^2 \mu(x) dS_x.$$

Hence, it is sufficient for us to establish the estimate (2.3a) and (2.3b). Next, we shall first prove (2.3a).

From Young’s inequality, we have the following estimate:

$$\int_{\omega_{\epsilon < \theta}} (\gamma \nabla \phi) \cdot (\gamma \nabla u_-^\epsilon) dx \leq \frac{1}{2} \int_{\omega_{\epsilon < \theta}} |\gamma \nabla \phi|^2 dx + \frac{1}{2} \int_{\omega_{\epsilon < \theta}} |\gamma \nabla u_-^\epsilon|^2 dx.$$

Since $\omega_\epsilon \rightarrow 0$ in $H^1(D)$, so $|\omega_{\epsilon > \theta}| \rightarrow 0$, when ϵ goes to zero. Therefore

$$\lim_{\epsilon \rightarrow 0} \int_{\omega_{\epsilon > \theta}} |\gamma \nabla \phi|^2 dx = 0.$$

By Hölder’s inequality, it holds that

$$\int_{\omega_{\epsilon > \theta}} (\gamma \nabla \phi) \cdot (\gamma \nabla u_-^\epsilon) dx \leq \left(\int_{\omega_{\epsilon > \theta}} |\gamma \nabla \phi|^2 dx \right)^{\frac{1}{2}} \left(\int_{\omega_{\epsilon > \theta}} |\gamma \nabla u_-^\epsilon|^2 dx \right)^{\frac{1}{2}},$$

which implies that

$$\lim_{\epsilon \rightarrow 0} \int_{\omega_{\epsilon > \theta}} (\gamma \nabla \phi) \cdot (\gamma \nabla u_-^\epsilon) dx = 0.$$

Since $u_-^\epsilon \rightarrow \tilde{u}_-$ in $H^1(D)$, we then arrive at the estimate (2.3a).

We proceed to establish the estimate (2.3b). From Young’s inequality, we have that

$$- \int_{\omega_{\epsilon > \theta}} (\gamma \nabla \omega_\epsilon) \cdot (\gamma \nabla u_-^\epsilon) \tilde{\phi} dx \leq \frac{1}{2} \int_{\omega_{\epsilon > \theta}} |\gamma \nabla \omega_\epsilon \tilde{\phi}|^2 dx + \frac{1}{2} \int_{\omega_{\epsilon > \theta}} |\gamma \nabla u_-^\epsilon|^2 dx,$$

which in turn implies

$$\begin{aligned} & \frac{1}{2} \int_{\omega_{\epsilon > \theta}} |\gamma \nabla u_-^\epsilon|^2 dx \\ & \geq - \left(\int_D - \int_{\omega_{\epsilon < \theta}} \right) (\gamma \nabla \omega_\epsilon) \cdot (\gamma \nabla u_-^\epsilon) \tilde{\phi} dx - \frac{1}{2} \left(\int_D - \int_{\omega_{\epsilon < \theta}} \right) |\gamma \nabla \omega_\epsilon \tilde{\phi}|^2 dx. \end{aligned} \tag{2.5}$$

Now we set

$$A := \int_{\omega_{\epsilon < \theta}} (\gamma \nabla \omega_\epsilon) \cdot (\gamma \nabla u_-^\epsilon) \tilde{\phi} dx, \tag{2.6a}$$

$$B := \int_{\omega_{\epsilon < \theta}} |\gamma \nabla \omega_\epsilon \tilde{\phi}|^2 dx. \tag{2.6b}$$

For A in (2.6a), we apply the Hölder’s inequality to obtain

$$\begin{aligned} A &\leq \left(\int_{\omega^\epsilon < \theta} |\gamma \nabla \omega_\epsilon \tilde{\phi}|^2 dx \right)^{\frac{1}{2}} \left(\int_{\omega^\epsilon < \theta} |\gamma \nabla u_-^\epsilon|^2 dx \right)^{\frac{1}{2}} \\ &\leq C \left(\int_{\omega^\epsilon < \theta} |\gamma \nabla \omega_\epsilon \tilde{\phi}|^2 dx \right)^{\frac{1}{2}}, \end{aligned} \tag{2.7}$$

where C is a constant depending only on a and b in the assumption (a1). Next, we prove that $B \leq C\theta$, which together with (2.7) readily yields (2.3b).

Let $\omega_\theta^\epsilon = (\theta - \omega_\epsilon)_+$. Then $\omega_\theta^\epsilon \in H_0^1(D)$ and $\omega_\theta^\epsilon \rightharpoonup \theta$ in $H_0^1(D)$. By integration by parts,

$$\lim_{\epsilon \rightarrow 0} \int_D (\gamma \nabla \omega_\epsilon) \cdot (\gamma \nabla (\omega_\theta^\epsilon \tilde{\phi}^2)) dx = 0,$$

where we have used the fact that

$$\nabla \cdot (\gamma^T \gamma \nabla \omega_\epsilon) = 0.$$

Thus

$$\lim_{\epsilon \rightarrow 0} \int_{D \cap \{\omega_\epsilon < \theta\}} |\gamma \nabla \omega_\epsilon|^2 \tilde{\phi}^2 dx = \lim_{\epsilon \rightarrow 0} \int_{D \cap \{\omega_\epsilon < \theta\}} (\gamma \nabla \omega_\epsilon) \cdot (\gamma \nabla \tilde{\phi}^2) \omega_\theta^\epsilon dx.$$

Since ω_ϵ is bounded in $H^1(D)$, we can apply the Hölder’s inequality to obtain

$$\int_{D \cap \{\omega_\epsilon < \theta\}} (\gamma \nabla \omega_\epsilon) \cdot (\gamma \nabla \tilde{\phi}^2) \omega_\theta^\epsilon dx \leq C \left(\int_D |\omega_\theta^\epsilon|^2 dx \right)^{\frac{1}{2}}.$$

Noting $\omega_\epsilon \rightharpoonup 0$ in $H^1(D)$, by the Sobolev embedding theorem [10], we can further obtain that

$$\lim_{\epsilon \rightarrow 0} \left(\int_D |\omega_\theta^\epsilon|^2 dx \right)^{\frac{1}{2}} = \theta.$$

Therefore, it holds that

$$B = \int_{\omega^\epsilon < \theta} |\gamma \nabla \omega_\epsilon \tilde{\phi}|^2 dx \leq C\theta,$$

which completes the proof of estimate (2.3b). The proof is completed. □

With Proposition 2.1 and Lemmas 2.1 and 2.2, one can handily prove Theorem 1.1 by following a similar argument to that in [23], which we only sketch as follows.

Proof of Theorem 1.1. The proof can be divided into three steps:

Step i. Choose an arbitrary function $v \in H^1(D)$ such that $v|_\Gamma = \psi(x)$, and we can guarantee that $v + (v - \varphi)_- \omega^\epsilon \in K$ under some assumption. Thus

$$J(v + (v - \varphi)_- \omega^\epsilon) \geq J(u^\epsilon).$$

Step ii. By Proposition 2.1 and Lemma 2.1, we can show

$$\lim_{\epsilon \rightarrow 0} J(v + (v - \varphi)_- \omega^\epsilon) \leq J_\mu(v),$$

and hence

$$\limsup_{\epsilon \rightarrow 0} J(u_\epsilon) \leq J_\mu(v).$$

Step iii. By Lemma 2.2, we have

$$\liminf_{\epsilon \rightarrow 0} J(u_\epsilon) \geq J_\mu(\tilde{u}),$$

and hence

$$J_\mu(\tilde{u}) \leq J_\mu(v).$$

This completes the proof. □

3 Construction of the corrector and its property

Let us first introduce the following Laplace-Beltrami operator associated with a Riemannian metric $g(x) = (g_{ij}(x))$, which is a symmetric positive definite matrix-valued function in local coordinates:

$$\Delta_g u = |g|^{-1/2} \sum_{i,j=1}^n \frac{\partial}{\partial x^i} \left(|g|^{1/2} g^{ij} \frac{\partial u}{\partial x^j} \right), \quad x = (x^j)_{j=1}^n \in \mathbb{R}^n, \tag{3.1}$$

where $G = \det(g)$ and $(g^{ij}) = (g_{ij})^{-1}$. By [20, 22], we know that there exists Green's function $\Phi_g(x, y)$ satisfying

$$\begin{cases} -\Delta_g \Phi_g(x, y) = \delta(x - y), & x, y \in \mathbb{R}^n, \\ \frac{\partial \Phi_g(x, y)}{\partial |x|} = \mathcal{O}(|x|^{-2}) & \text{as } |x| \rightarrow \infty, \end{cases}$$

and possessing the following expansion in a small open neighborhood of y :

$$\begin{aligned} \Phi_g(x, y) &\sim C_1(x) d_g(x, y)^{2-n} + C_2(x) d_g(x, y)^{3-n} + \dots \\ &\quad + C_{n-2}(x) d_g(x, y)^{-1} + C_n(x) d_g(x, y) + \dots, \\ \nabla_x \Phi_g(x, y) &\sim C'_1(x) d_g(x, y)^{1-n} V_{x,y} + C'_2(x) d_g(x, y)^{2-n} V_{x,y} + \dots, \end{aligned}$$

where $d_g(x, y)$ is the Riemannian distance function and $C_j(x) \in C^\infty$ for all $j \in \mathbb{N}$. In what follows, we treat γ equivalent to $|g|^{1/2}g^{-1}$.

Next, we begin to construct corrector ω_ϵ . Define

$$\omega_{\epsilon,k}(x) := \begin{cases} 1, & d_g(x, x_{\epsilon,k}) \leq r_{\epsilon,k}, \\ \frac{\Phi_g(x, x_{\epsilon,k}) - \epsilon^{2-n}}{r_{\epsilon,k}^{2-n} - \epsilon^{2-n}}, & r_{\epsilon,k} < d_g(x, x_{\epsilon,k}) < \epsilon, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

Evidently, the support of $\omega_{\epsilon,k}$ is situated in the ball $B_\epsilon(x_{\epsilon,k})$, and these balls are disjoint through our assumptions. Thus we can select our corrector as follows :

$$\omega_\epsilon := \sum_k \omega_{\epsilon,k}.$$

It is easy to see that

$$\|\omega_\epsilon\|_{L^\infty(D)} \leq C.$$

We proceed to verify the conditions (2.1d) and (2.2) in Proposition 2.1. Firstly, we can obtain the following result :

$$\lim_{\epsilon \rightarrow 0} \int_D |\omega_\epsilon|^2 dx = 0. \tag{3.3}$$

In fact, one can deduce that

$$\begin{aligned} & \int_{B_\epsilon} |\omega_\epsilon|^2 dx = |B_{r_\epsilon}| + \int_{B_\epsilon \setminus B_{r_\epsilon}} |\omega_\epsilon|^2 dx \\ & \leq |B_{r_\epsilon}| + \frac{C}{(r_\epsilon^{2-n} - \epsilon^{2-n})^2} \int_{B_\epsilon \setminus B_{r_\epsilon}} \left(d_g(x, 0)^{2-n} - \epsilon^{2-n} + \sum_{j=3, j \neq n}^\infty d_g(x, 0)^{j-n} \right)^2 dx \\ & \leq |B_{r_\epsilon}| + \frac{C(n)}{(r_\epsilon^{2-n} - \epsilon^{2-n})^2} \int_{r_\epsilon}^\epsilon \left((r^{2-n} - \epsilon^{2-n})^2 r^{n-1} + \sum_{j=3, j \neq n}^\infty r^{2j-2n} r^{n-1} \right. \\ & \quad \left. + (r^{2-n} - \epsilon^{2-n}) \sum_{j=3, j \neq n}^\infty r^{j-n} r^{n-1} \right) dr \\ & \leq |B_{r_\epsilon}| + C(n) \mathcal{O}(\epsilon^n) + \frac{C(n)}{(r_\epsilon^{2-n} - \epsilon^{2-n})^2} \int_{r_\epsilon}^\epsilon \left(\sum_{j=3}^\infty r^{2j-n-1} + (r^{2-n} - \epsilon^{2-n}) \sum_{j=3}^\infty r^{j-1} \right) dr \\ & \leq |B_{r_\epsilon}| + C(n) \left(\mathcal{O}(\epsilon^n) + \sum_{j=4}^\infty \max \left\{ \mathcal{O}(\epsilon^{n+j}), \mathcal{O}(\epsilon^{\frac{(n-1)(n+j-2)}{n-2}}) \right\} + \sum_{j=1}^\infty \mathcal{O}(\epsilon^{2n+j}) \right) \end{aligned}$$

$$\begin{aligned} &\leq C(n) \left(\mathcal{O}(\epsilon^n) + \max \left\{ \mathcal{O}(\epsilon^{n+3}), \mathcal{O}\left(\epsilon^{\frac{n(n-1)}{n-2}}\right) \right\} + \mathcal{O}(\epsilon^{2n}) \right) \\ &\leq C(n) \mathcal{O}(\epsilon^n), \end{aligned}$$

which readily yields (3.3).

Secondly, we can estimate the term $\|\nabla\omega_\epsilon\|_{L^2(D)}$ as follows:

$$\begin{aligned} \int_{B_\epsilon} |\nabla\omega_\epsilon|^2 dx &\leq \frac{C}{(r_\epsilon^{2-n} - \epsilon^{2-n})^2} \int_{B_\epsilon \setminus B_{r_\epsilon}} \sum_{j=1, j \neq n}^\infty d_g(x, 0)^{2j-2n} dx \\ &\leq \frac{C(n)}{(r_\epsilon^{2-n} - \epsilon^{2-n})^2} \int_{r_\epsilon}^\epsilon \left(r^{1-n} + \sum_{j=2}^\infty r^{2j-n-1} \right) dr \\ &\leq C(n) \left(\mathcal{O}(\epsilon^{n-1}) + \sum_{j=2}^\infty \max \left\{ \mathcal{O}(\epsilon^{n+j}), \mathcal{O}\left(\epsilon^{\frac{(n+j-2)(n-1)}{n-2}}\right) \right\} \right) \\ &\leq C(n) \left(\mathcal{O}(\epsilon^{n-1}) + \max \left\{ \mathcal{O}(\epsilon^{n+1}), \mathcal{O}\left(\epsilon^{\frac{(n-1)^2}{n-2}}\right) \right\} \right) \\ &\leq C(n) \mathcal{O}(\epsilon^{n-1}), \end{aligned}$$

which in turn implies that

$$\int_D |\nabla\omega_\epsilon|^2 dx \leq C. \tag{3.4}$$

Finally, we can combine the estimates (3.3) and (3.4) to derive that

$$\omega_\epsilon \rightharpoonup 0 \quad \text{in } H^1(D),$$

which readily verifies the condition (2.1d).

4 Limiting property of the corrector

To complete the proof of Proposition 2.1, it suffices to verify the inequality (2.2). For simplicity, we write

$$B_{\epsilon,k} := B_\epsilon(x_{\epsilon,k}) \setminus B_{r_{\epsilon,k}}(x_{\epsilon,k}), \quad D_{\epsilon,k} := (B_\epsilon(x_{\epsilon,k}) \setminus B_{r_{\epsilon,k}}(x_{\epsilon,k})) \cap D.$$

If ϵ is sufficiently small, we clearly have

$$\int_D (\gamma \nabla \omega_\epsilon) \cdot (\gamma \nabla v_\epsilon) \phi dx = \sum_k \int_{D_{\epsilon,k}} (\gamma \nabla \omega_\epsilon) \cdot (\gamma \nabla v_\epsilon) \phi dx.$$

Integrating by parts, it holds that

$$\begin{aligned}
 & \int_{D_{\epsilon,k}} (\gamma \nabla \omega_\epsilon) \cdot (\gamma \nabla v_\epsilon) \phi dx \\
 &= \int_{D_{\epsilon,k}} \nabla \cdot (\gamma^T \gamma \nabla \omega_\epsilon v_\epsilon \phi) dx - \int_{D_{\epsilon,k}} (\gamma \nabla \omega_\epsilon) \cdot (\gamma \nabla \phi) v_\epsilon dx \\
 &= \int_{\partial D_{\epsilon,k}} (\gamma^T \gamma \nabla \omega_\epsilon) \cdot \vec{\nu} v_\epsilon \phi dS_x - \int_{D_{\epsilon,k}} (\gamma \nabla \omega_\epsilon) \cdot (\gamma \nabla \phi) v_\epsilon dx \\
 &= \int_{\partial D_{\epsilon,k}} (\gamma \nabla \omega_\epsilon) \cdot (\gamma \vec{\nu}) v_\epsilon \phi dS_x - \int_{D_{\epsilon,k}} (\gamma \nabla \omega_\epsilon) \cdot (\gamma \nabla \phi) v_\epsilon dx, \tag{4.1}
 \end{aligned}$$

where we make use of the following fact :

$$\nabla \cdot (\gamma^T \gamma \nabla \omega_\epsilon) = 0, \quad \forall x \in D_{\epsilon,k}.$$

In addition, we can get the following result :

$$\lim_{\epsilon \rightarrow 0} \sum_k \int_{D_{\epsilon,k}} (\gamma \nabla \omega_\epsilon) \cdot (\gamma \nabla \phi) v_\epsilon dx = 0. \tag{4.2}$$

In fact, by direct calculations, we have

$$\begin{aligned}
 \int_{B_\epsilon} |\gamma \nabla \omega_\epsilon| dx &\leq \frac{C}{r_\epsilon^{2-n} - \epsilon^{2-n}} \int_{B_\epsilon \setminus B_{r_\epsilon}} \sum_{j=1, j \neq n}^{\infty} d_g(x, 0)^{j-n} dx \\
 &\leq \frac{C(n)}{r_\epsilon^{2-n} - \epsilon^{2-n}} \int_{r_\epsilon}^{\epsilon} \sum_{j=1}^{\infty} r^{j-1} dr \\
 &\leq C(n) \mathcal{O}(\epsilon^n) + C(n) \sum_{j=1}^{\infty} \mathcal{O}(\epsilon^{n+j}), \\
 &\leq C(n) \mathcal{O}(\epsilon^n),
 \end{aligned}$$

and hence

$$\int_D |\gamma \nabla \omega_\epsilon| |\gamma \nabla \phi| v_\epsilon dx \leq C \int_D |\gamma \nabla \omega_\epsilon| dx \leq C\epsilon,$$

where C depends only on n , a and b . Consequently, we have established (4.2). This, together with (4.1), inspires us to only consider the following integral

$$\int_{\partial D_{\epsilon,k}} (\gamma \nabla \omega_\epsilon) \cdot (\gamma \vec{\nu}) v_\epsilon \phi dS_x$$

$$\begin{aligned}
 &= \left(\int_{\partial B_{r_\epsilon}(x_{\epsilon,k}) \cap D} + \int_{\partial D \cap B_{\epsilon,k}} + \int_{\partial B_\epsilon(x_{\epsilon,k}) \cap D} \right) (\gamma \nabla \omega_\epsilon) \cdot (\gamma \vec{\nu}) v_\epsilon \phi dS_x \\
 &=: E_1 + E_2 + E_3.
 \end{aligned} \tag{4.3}$$

Noticing that

$$\begin{aligned}
 &\int_{\partial B_{r_\epsilon} \cap D} (\gamma \nabla \omega_\epsilon) \cdot (\gamma \vec{\nu}) v_\epsilon \phi dS_x \\
 &\geq \int_{\partial B_{r_\epsilon} \cap D} \frac{\sum_{i=1}^n C'_i r_\epsilon^{i-n}}{r_\epsilon^2 (r_\epsilon^{2-n} - \epsilon^{2-n})} (\gamma x)^T (\gamma x) v_\epsilon \phi dS_x \\
 &\geq \frac{-C \|v_\epsilon \phi\|_{L^\infty}}{r_\epsilon^{2-n} - \epsilon^{2-n}} \int_{\partial B_{r_\epsilon} \cap D} \sum_{i=1}^{+\infty} r_\epsilon^{i-n} dS_x \\
 &\geq -\frac{C \epsilon^{n-1}}{1-r_\epsilon},
 \end{aligned} \tag{4.4}$$

where $\{C'_i\}_{i=1}^\infty$ are bounded. Hence, we see that

$$\lim_{\epsilon \rightarrow 0} \sum_k \int_{\partial B_{r_\epsilon}(x_{\epsilon,k}) \cap D} (\gamma \nabla \omega_\epsilon) \cdot (\gamma \vec{\nu}) v_\epsilon \phi dS_x \geq 0.$$

In the following, we need to estimate E_2 and E_3 . For the estimation of E_2 , the key idea is to make a local flattening of the boundary ∂D because of $\partial D \in C^{1,\alpha}$. More precisely, we have the following lemma.

Lemma 4.1. *Let ω_ϵ be defined in (3.2). Then*

$$\lim_{\epsilon \rightarrow 0} \sum_k \int_{\partial D \cap B_{\epsilon,k}} (\gamma \nabla \omega_\epsilon) \cdot (\gamma \vec{\nu}) v_\epsilon \phi dS_x = 0.$$

Proof. Without loss of generality, we may suppose that $x_{\epsilon,k}$ is the origin. Since $\partial D \in C^{1,\alpha}$, then near the origin, ∂D can be given by

$$x_n = \rho(x'),$$

where ρ is $C^{1,\alpha}$ near the origin in \mathbb{R}^{n-1} and $|\rho(x')| \leq C|x'|^{1+\alpha}$ for $|x'|$ small enough. Furthermore, there holds

$$D \cap B_\epsilon \subset \{x_n > \rho(x')\} \quad \text{for } \epsilon \text{ small.}$$

Let $\nu(x')$ be the outward unit normal vector on ∂D at $(x', \rho(x'))$ near the origin. Then $\nu \in C^\alpha$ near the origin in \mathbb{R}^{n-1} . Substituting by variables, we have

$$\begin{aligned} & \int_{\partial D \cap (B_\epsilon \setminus B_{r_\epsilon})} (\gamma \nabla \omega_\epsilon) \cdot (\gamma \vec{\nu}) v_\epsilon \phi dS_x \\ &= \int_{(B_\epsilon \setminus B_{r_\epsilon})} (\gamma \nabla \omega_\epsilon(x', \rho(x'))) \cdot (\gamma \vec{\nu}(x')) v_\epsilon \phi \sqrt{1 + |\nabla \rho(x')|^2} dx'. \end{aligned} \tag{4.5}$$

Next, we split the term $(\gamma \nabla \omega_\epsilon(x', \rho(x'))) \cdot (\gamma \vec{\nu}(x'))$ into three parts :

$$\begin{aligned} & (\gamma \nabla \omega_\epsilon(x', \rho(x'))) \cdot (\gamma \vec{\nu}(x')) \\ &= (\gamma \nabla \omega_\epsilon(x', 0)) \cdot (\gamma \vec{\nu}(0)) + (\gamma \nabla \omega_\epsilon(x', \rho(x')) - \gamma \nabla \omega_\epsilon(x', 0)) \cdot (\gamma \vec{\nu}(0)) \\ & \quad + (\gamma \nabla \omega_\epsilon(x', \rho(x'))) \cdot (\gamma \vec{\nu}(x') - \gamma \vec{\nu}(0)) \\ &=: A_1 + A_2 + A_3. \end{aligned} \tag{4.6}$$

We note that $A_1 = 0$ by the construction of ω_ϵ . Hence, we only need to focus on the estimates of A_2 and A_3 .

To facilitate the estimate of the term A_2 , we define

$$A_{2,1} := \sum_{i=1}^{n-1} \left| \frac{\partial \omega_\epsilon}{\partial x_i}(x', \rho(x')) - \frac{\partial \omega_\epsilon}{\partial x_i}(x', 0) \right|, \tag{4.7a}$$

$$A_{2,2} := \left| \frac{\partial \omega_\epsilon}{\partial x_n}(x', \rho(x')) - \frac{\partial \omega_\epsilon}{\partial x_n}(x', 0) \right|. \tag{4.7b}$$

Then we have by direct computations that

$$\begin{aligned} A_{2,1} &:= \sum_{i=1}^{n-1} \left| \frac{\partial \omega_\epsilon}{\partial x_i}(x', \rho(x')) - \frac{\partial \omega_\epsilon}{\partial x_i}(x', 0) \right| \\ &\leq \sum_{i=1}^{n-1} \sup_{0 \leq x_n \leq \rho(x')} \left| \frac{\partial^2 \omega_\epsilon}{\partial x_i \partial x_n}(x', x_n) \rho(x') \right| \\ &\leq \frac{C}{r_\epsilon^{2-n} - \epsilon^{2-n}} \sum_{i=1}^{n-1} \sup_{0 \leq x_n \leq \rho(x')} \sum_{j=1}^{\infty} \left\{ d_g(x, 0)^{j+1-n} + (|x_n| + |x_i|) d_g(x, 0)^{j-1-n} \right. \\ & \quad \left. + |x_n| |x_i| d_g(x, 0)^{j-3-n} \right\} |\rho(x')| \\ &\leq \frac{C}{r_\epsilon^{2-n} - \epsilon^{2-n}} \sum_{i=1}^{n-1} \sup_{0 \leq x_n \leq \rho(x')} \left\{ |x'|^{1-n} + M + (|x_n| + |x_i|) (|x'|^{-n-1} + M) \right. \\ & \quad \left. + |x_n| |x_i| (|x'|^{-3-n} + M) \right\} |\rho(x')| \left(M = \frac{1}{1 - (|x'| + |x'|^{1+\alpha})} \right) \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{C(n)}{r_\epsilon^{2-n} - \epsilon^{2-n}} \left\{ |x'|^{1-n} + M + (|x'|^{1+\alpha} + |x'|)(|x'|^{-n-1} + M) \right. \\
 &\quad \left. + |x'|^{2+\alpha}(|x'|^{-3-n} + M) \right\} |x'|^{1+\alpha} \\
 &\leq \frac{C(n)}{r_\epsilon^{2-n} - \epsilon^{2-n}} \left\{ |x'|^\alpha(1 + |x'| + |x'|^{1+\alpha} + |x'|^{\alpha+2}) + |x'|^{\alpha-n+2} + |x'|^{\alpha-n+1} \right. \\
 &\quad \left. + |x'|^{1+2\alpha-n} + |x'|^{2\alpha-n} \right\}. \tag{4.8}
 \end{aligned}$$

Similarly, we can show that

$$\begin{aligned}
 A_{2,2} &:= \left| \frac{\partial \omega_\epsilon}{\partial x_n}(x', \rho(x')) - \frac{\partial \omega_\epsilon}{\partial x_n}(x', 0) \right| \\
 &\leq \frac{C}{r_\epsilon^{2-n} - \epsilon^{2-n}} \left\{ |x'|^\alpha(1 + |x'|^{1+\alpha} + |x'|^{2\alpha+2}) \right. \\
 &\quad \left. + |x'|^{\alpha-n+2} + |x'|^{1+2\alpha-n} + |x'|^{3\alpha-n} \right\}. \tag{4.9}
 \end{aligned}$$

Then we have by combining the two estimates (4.8) and (4.9) that

$$\begin{aligned}
 |A_2| &\leq \frac{C}{r_\epsilon^{2-n} - r^{2-n}} \left\{ |x'|^\alpha(1 + |x'| + |x'|^{1+\alpha} + |x'|^{2\alpha-n} + |x'|^{\alpha+2} + |x'|^{2+2\alpha}) \right. \\
 &\quad \left. + |x'|^{\alpha-n+1} + |x'|^{\alpha-n+2} + |x'|^{2\alpha-n+1} + |x'|^{3\alpha-n} \right\}, \tag{4.10}
 \end{aligned}$$

where C depends only on n, a, b , and $|C'_i|_{L^\infty(D)}$, ($i = 1, 2, \dots$).

For the term A_3 , let us first recall that $\partial D \in C^{1,\alpha}$, then we will select ϵ sufficiently small such that $|\nabla \rho| \leq 1$ for $|x'| < \epsilon$, and

$$|\gamma(\vec{\nu}(x') - \vec{\nu}(0))| \leq C|x'|^\alpha.$$

Hence

$$|A_3| \leq C|x'|^\alpha |\gamma \nabla \omega_\epsilon(x', \rho(x'))|. \tag{4.11}$$

In view of (4.6), (4.10) and (4.11), it concludes that

$$\begin{aligned}
 &\int_{(B_\epsilon \setminus B_{r_\epsilon})} (\gamma \nabla \omega_\epsilon(x', \rho(x'))) \cdot (\gamma \vec{\nu}(x')) v_\epsilon \phi \sqrt{1 + |\nabla \rho(x')|^2} dx' \\
 &\leq 2 \|\phi v_\epsilon\|_{L^\infty} \int_{(B_\epsilon \setminus B_{r_\epsilon})} |(\gamma \nabla \omega_\epsilon(x', \rho(x'))) \cdot (\gamma \vec{\nu}(x'))| dx' \\
 &\leq 2 \|\phi v_\epsilon\|_{L^\infty} \int_{(B_\epsilon \setminus B_{r_\epsilon})} |\gamma \nabla \omega_\epsilon(x', \rho(x')) - \gamma \nabla \omega_\epsilon(x', 0)| + C |\gamma \nabla \omega_\epsilon(x', \rho(x'))| |x'|^\alpha dx'
 \end{aligned}$$

$$\begin{aligned}
 &\leq 2\|\phi v_\epsilon\|_{L^\infty} \int_{(B_\epsilon \setminus B_{r_\epsilon})} \left\{ |x'|^\alpha (1 + |x'| + |x'|^{1+\alpha} + |x'|^{\alpha+2} + |x'|^{2+2\alpha}) + |x'|^{\alpha-n+1} \right. \\
 &\quad \left. + |x'|^{\alpha-n+2} + |x'|^{2\alpha-n} + |x'|^{2\alpha-n+1} + |x'|^{3\alpha-n} \right\} dx' \\
 &\leq \|\phi v_\epsilon\|_{L^\infty} \left\{ \mathcal{O}(\epsilon^{\alpha+2n-2})(1+\epsilon) + \mathcal{O}(\epsilon^{2\alpha+2n-1})(1+\epsilon+\epsilon^{\alpha+1}) + \mathcal{O}(\epsilon^{\alpha+n-1})(1+\epsilon) \right. \\
 &\quad \left. + \mathcal{O}(\epsilon^{2\alpha+n-2})(1+\epsilon+\epsilon^\alpha) \right\} \\
 &\leq \|\phi v_\epsilon\|_{L^\infty} \mathcal{O}(\epsilon^{2\alpha+n-2}), \tag{4.12}
 \end{aligned}$$

which, together with (4.5) and $\alpha \in (2^{-1}, 1)$, yields that

$$\lim_{\epsilon \rightarrow 0} \sum_k \int_{\partial D \cap B_{\epsilon,k}} (\gamma \nabla \omega_\epsilon) \cdot (\gamma \vec{\nu}) v_\epsilon \phi dS_x = 0.$$

Hence we complete the proof of Lemma 4.1. □

The next lemma tells us the estimate of E_3 . It is worth noting that the construction of the auxiliary function $q_\epsilon(x)$ plays an important role in its estimation.

Lemma 4.2. *Suppose that ω_ϵ satisfies the definition of the above section. Then there holds*

$$\lim_{\epsilon \rightarrow 0} \sum_k \int_{\partial B_\epsilon(x_{\epsilon,k}) \cap D} (\gamma \nabla \omega_\epsilon) \cdot (\gamma \vec{\nu}) v_\epsilon \phi dS_x \geq - \int_\Sigma \phi v \mu(x) dS_x,$$

with equality held if $v_\epsilon|_{T_\epsilon} = 0$.

Proof. Suppose that ϵ is sufficiently small. By direct calculations, we have

$$(\gamma \nabla \omega_\epsilon) \cdot (\gamma \vec{\nu}) = \frac{\sum_{i=1}^\infty \tilde{C}_i \epsilon^{i-n}}{\epsilon^2 (r_\epsilon^{2-n} - \epsilon^{2-n})} (\gamma x)^T (\gamma x) = \frac{\sum_{i=1}^\infty \tilde{C}_i \epsilon^{i-1}}{\epsilon^2 (\tilde{r}_\epsilon^{2-n} - \epsilon)} (\gamma x)^T (\gamma x)$$

on ∂B_ϵ , where $r_\epsilon = \tilde{r}_\epsilon \epsilon^{\frac{n-1}{n-2}}$ and $\{\tilde{C}_i\}_{i=1}^\infty$ are bounded.

Motivated by Cioranescu and Murant [6, 7], we consider the following auxiliary function

$$q_\epsilon(x) := \begin{cases} \frac{\sum_{i=1}^\infty \tilde{C}_i \epsilon^{i-1}}{2\epsilon(\tilde{r}_\epsilon^{2-n} - \epsilon)} (d_g^2(x, 0) - \epsilon^2), & d_g(x, 0) \leq \epsilon, \\ 0, & d_g(x, 0) > \epsilon, \end{cases}$$

which gives that

$$(\gamma \nabla q_\epsilon) \cdot (\gamma \vec{\nu}) = (\gamma \nabla \omega_\epsilon) \cdot (\gamma \vec{\nu}) \quad \text{on } \partial B_\epsilon, \tag{4.13}$$

and

$$\lim_{\epsilon \rightarrow 0} \int_D |\nabla q_\epsilon(x)|^2 dx = 0. \tag{4.14}$$

In fact, (4.13) is obvious. In the following, we need to show (4.14). In fact, direct calculation yields that

$$\begin{aligned} \int_{B_\epsilon} |\nabla q_\epsilon(x)|^2 dx &= \int_{B_\epsilon} \left(\frac{\sum_{i=1}^\infty \tilde{C}_i \epsilon^{i-1}}{\tilde{r}_\epsilon^{2-n} - \epsilon} \right)^2 \frac{x^2}{\epsilon^2} dx \\ &\leq C(n) \left(\frac{\sum_{i=1}^\infty \tilde{C}_i \epsilon^{i-1}}{(\tilde{r}_\epsilon^{2-n} - \epsilon)\epsilon} \right)^2 \int_0^\epsilon r^{n+1} dr \\ &\leq C(n) \frac{(\sum_{i=1}^\infty \tilde{C}_i \epsilon^{i-1})^2}{(\tilde{r}_\epsilon^{2-n} - \epsilon)^2} \epsilon^n \\ &\leq C(n) \left(\frac{1}{(1-\epsilon)(\tilde{r}_\epsilon^{2-n} - \epsilon)} \right)^2 \epsilon^n (c_1 \leq \tilde{r}_\epsilon \leq c_2) \\ &\leq C(n) \mathcal{O}(\epsilon^n), \end{aligned}$$

which verifies (4.14).

Integrating by parts, we see that

$$\begin{aligned} &\int_D \phi v_\epsilon \nabla \cdot (\gamma^T \gamma \nabla q_\epsilon) dx \\ &= - \int_D (\gamma \nabla(\phi v_\epsilon)) \cdot (\gamma \nabla q_\epsilon(x)) dx + \sum_k \int_{\partial(B_\epsilon(x_{\epsilon,k}) \cap D)} (\gamma \nabla q_\epsilon) \cdot (\gamma \vec{\nu}) \phi v_\epsilon dS_x. \end{aligned} \tag{4.15}$$

Then by applying Hölder’s inequality to the first term of (4.15), we have

$$\int_D (\gamma \nabla(\phi v_\epsilon)) \cdot (\gamma \nabla q_\epsilon) dx \leq C(n) \|\gamma \nabla(\phi v_\epsilon)\|_{L^\infty(D)} \int_D |\gamma \nabla q_\epsilon|^2 dx,$$

which means that (by (4.14))

$$\lim_{\epsilon \rightarrow 0} \int_D (\gamma \nabla(\phi v_\epsilon)) \cdot (\gamma \nabla q_\epsilon(x)) dx = 0.$$

Therefore it suffices for us to only study the second term of (4.15),

$$\begin{aligned}
 & \sum_k \int_{\partial(B_\epsilon(x_{\epsilon,k}) \cap D)} (\gamma \nabla q_\epsilon) \cdot (\gamma \vec{\nu}) \phi v_\epsilon dS_x \\
 &= \sum_k \left(\int_{\partial B_\epsilon(x_{\epsilon,k}) \cap D} + \int_{\partial D \cap B_\epsilon(x_{\epsilon,k})} \right) (\gamma \nabla q_\epsilon) \cdot (\gamma \vec{\nu}) \phi v_\epsilon dS_x \\
 &=: E_{3,1} + E_{3,2}.
 \end{aligned} \tag{4.16}$$

For the term $E_{3,2}$, we can use a similar argument to that for Lemma 4.1 to obtain

$$\begin{aligned}
 & \int_{\partial D \cap B_\epsilon} (\gamma \nabla q_\epsilon) \cdot (\gamma \vec{\nu}) \phi v_\epsilon dS_x \\
 & \leq C \|\phi v_\epsilon\|_{L^\infty} \int_{\partial D \cap B_\epsilon} \frac{\sum_{i=1}^\infty \tilde{C}_i \epsilon^{i-1}}{(\tilde{r}_\epsilon^{2-n} - \epsilon)\epsilon} |x \cdot \vec{\nu}| dS_x \\
 & \leq C \|\phi v_\epsilon\|_{L^\infty} \int_{\partial D \cap B_\epsilon} \frac{\sum_{i=1}^\infty \tilde{C}_i \epsilon^{i-1}}{(\tilde{r}_\epsilon^{2-n} - \epsilon)\epsilon} |x'|^{1+\alpha} dS_x \\
 & \leq C \|\phi v_\epsilon\|_{L^\infty} \int_{B_\epsilon} \frac{\sum_{i=1}^\infty \tilde{C}_i \epsilon^{i-1}}{(\tilde{r}_\epsilon^{2-n} - \epsilon)\epsilon} |x'|^{1+\alpha} dx' \\
 & \leq C \|\phi v_\epsilon\|_{L^\infty} \frac{\sum_{i=1}^\infty \tilde{C}_i \epsilon^{i-1}}{(\tilde{r}_\epsilon^{2-n} - \epsilon)\epsilon} \int_0^\epsilon r^{n+\alpha-1} dr \\
 & \leq C \|\phi v_\epsilon\|_{L^\infty} \frac{\sum_{i=1}^\infty \tilde{C}_i \epsilon^{i-1}}{\tilde{r}_\epsilon^{2-n} - \epsilon} \epsilon^{n+\alpha-1} \\
 & \leq \frac{C}{(\tilde{r}_\epsilon^{2-n} - \epsilon)(1-\epsilon)} \epsilon^{n+\alpha-1} \quad (c_1 \leq \tilde{r}_\epsilon \leq c_2) \\
 & = C \mathcal{O}(\epsilon^{n+\alpha-1}),
 \end{aligned} \tag{4.17}$$

where C is a positive constant depending only on $n, a, b, \|\phi v_\epsilon\|_{L^\infty(D)}$ and $\|C'_i\|_{L^\infty(D)}$.

Thus

$$\lim_{\epsilon \rightarrow 0} \sum_k \int_{\partial D \cap B_\epsilon(x_{\epsilon,k})} (\gamma \nabla q_\epsilon) \cdot (\gamma \vec{\nu}) \phi v_\epsilon dS_x = 0.$$

For the term $E_{3,1}$, returning now to (4.15), we need to compute

$$\begin{aligned}
 & \int_{D \cap B_\epsilon} \phi v_\epsilon \nabla \cdot (\gamma^T \gamma \nabla q_\epsilon) dx \\
 &= \int_{D \cap B_\epsilon} \frac{\sum_{i=1}^\infty \tilde{C}_i \epsilon^{i-1}}{(\tilde{r}_\epsilon^{2-n} - \epsilon)\epsilon} \nabla \cdot (\gamma^T \gamma x) \phi v_\epsilon dx
 \end{aligned}$$

$$= - \int_D \frac{(\sum_{i=1}^{\infty} \tilde{C}_i \epsilon^{i-1}) \tilde{r}_\epsilon^{n-2}}{\epsilon \tilde{r}_\epsilon^{n-2} - 1} \nabla \cdot (\gamma^T \gamma x) \frac{1}{\epsilon} \chi_{B_\epsilon}(x) \phi v_\epsilon dx. \quad (4.18)$$

By (4.13), (4.15), (4.18) and the definition of $\mu_\epsilon(x)$, we obtain

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \sum_k \int_{\partial B_\epsilon(x_{\epsilon,k}) \cap D} (\gamma \nabla \omega_\epsilon) \cdot (\gamma \vec{\nu}) \phi v_\epsilon dS_x \\ &= \lim_{\epsilon \rightarrow 0} \sum_k \int_{\partial B_\epsilon(x_{\epsilon,k}) \cap D} (\gamma \nabla q_\epsilon) \cdot (\gamma \vec{\nu}) \phi v_\epsilon dS_x \\ &\geq - \lim_{\epsilon \rightarrow 0} \int_D \phi v_\epsilon \mu_\epsilon(x) dx. \end{aligned} \quad (4.19)$$

From our assumption, we have

$$\lim_{\epsilon \rightarrow 0} \int_D \phi v_\epsilon \mu_\epsilon(x) dx = \int_\Sigma \phi v \mu(x) dS_x.$$

Therefore

$$\lim_{\epsilon \rightarrow 0} \sum_k \int_{\partial B_\epsilon(x_{\epsilon,k}) \cap D} (\gamma \nabla \omega_\epsilon) \cdot (\gamma \vec{\nu}) \phi v_\epsilon dS_x \geq - \int_\Sigma \phi v \mu(x) dS_x.$$

This finishes the proof of Lemma 4.2. \square

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