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Linearly Compact Difference Scheme for the Two-Dimensional Kuramoto-Tsuzuki Equation with the Neumann Boundary Condition

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This work is dedicated to Prof. Hai-Wei Sun on the occasion of his 60th birthday

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Abstract. In this paper, we analyze and test a high-order compact difference scheme numerically for solving a two-dimensional nonlinear Kuramoto-Tsuzuki equation under the Neumann boundary condition. A three-level average technique is utilized, thereby leading to a linearized difference scheme. The main work lies in the pointwise error estimate in H^2-norm. The optimal fourth-order convergence order is proved in combination of induction, the energy method and the embedded inequality. Moreover, we establish the stability of the difference scheme with respect to the initial value under very mild condition, however, does not require any step ratio restriction. Extensive numerical examples with/without exact solutions under diverse cases are implemented to validate the theoretical results.

AMS subject classifications: 65M06, 65M12, 65M15

Key words: Kuramoto-Tsuzuki equation, compact difference scheme, pointwise error estimate, stability, numerical simulation.

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1 Introduction

In this paper, we will study a high-order compact difference scheme for the initial-boundary value problem of the two-dimensional nonlinear Kuramoto-Tsuzuki (KT) equation in the form of

$$\begin{cases} u_t=(1+\mathrm{i}c_1)\Delta u+\gamma u-(1+\mathrm{i}c_2)|u|^2u, & (x,y)\in\Omega, \quad 0<t\leq T, & (1.1\mathrm{a}) \\ u(x,y,0)=\varphi(x,y), & (x,y)\in\bar{\Omega}, \quad 0<t\leq T, & (1.1\mathrm{b}) \\ \dfrac{\partial u}{\partial \boldsymbol{\nu}}=0, & 0<t\leq T, & (1.1\mathrm{c}) \end{cases}$$

where u is an unknown complex function. $\mathrm{i}=\sqrt{-1}$, c_1 and c_2 are two real constants, which could characterize linear and nonlinear dispersion respectively, see e.g., [2]. γ is a general parameter, which controls the degree of aggregation of solutions. The calculated domain is on $\Omega=(0,L_1)\times(0,L_2)$, and $\boldsymbol{\nu}$ is the unit normal vector of the boundary Ω. $\partial\Omega$ is the boundary of the domain. $\varphi(x,y)$ is a given function.

The KT equation [6, 7] describes the behavior of two branches near the bifurcation point. Many efforts have been made to develop highly effective algorithms for the KT equation in one dimension. For example, Tsertsvadze [18] applied Crank-Nicolson method to establish a nonlinear difference scheme for solving the one-dimensional KT equation with the convergence order $\mathcal{O}(h^{\frac{3}{2}})$ in the sense of discrete L^2-norm. Ivanauskas [5] investigated an effective implicit Crank-Nicolson type weighted scheme for the KT equation and the convergence was proved. Sun [17] constructed a linearized three-level difference scheme, which can be solved by the double-sweep method and proved that it is uniquely solvable and convergent. Sun [13,14,16] further developed several new second-order difference schemes and made detailed analysis at length. Štikonas [12] discussed the root condition of a finite difference scheme for the KT equation. Omrani [11] analyzed the convergence of Galerkin method for the KT equation. Wang et al. [19, 20] respectively used semi-explicit difference scheme and nonlinear difference scheme for solving the KT equation. Dong [2] gave a fourth-order split-step pseudospectral scheme and Hu et al. [4] first proposed several fourth-order compact difference schemes for solving the KT equation.

As far as we know, no research work has been done about the numerical solutions of the high-dimensional KT equation under the Neumann boundary condition. Therefore, it is necessary to develop effective numerical algorithms for the KT equation in high dimension. The studies that have been done for high-dimensional KT equation so far include the following two work. One of them dues to Li et al. [9], who discussed a type of the high-dimensional KT equation with Dirichlet boundary condition by Galerkin finite element method and the optimal error estimates are obtained in L^2-norm. The other one is done by Xu and Chang [21]. They studied three classes of numerical schemes for the two-dimensional Ginzburg-Landau equation with the Neumann boundary condition, which involves the two-dimensional KT equation as its special case. However, both schemes are second-order or third-order convergent in L^2-norm. The pointwise error estimates in H^2-norm for the KT equation with the Neumann boundary condition in two dimension still remain unsolved. This is the main motivation to start this paper.

The main work aims at that we prove the pointwise error estimate of the present difference scheme for the two-dimensional KT equation with the Neumann boundary condition for the first time. Furthermore, the optimal convergence order $\mathcal{O}(\tau^2+h_1^4+h_2^4)$ in the maximum norm is obtained, where τ and h_1, h_2 denote the temporal step size and spatial step sizes, respectively. In addition, we prove the stability based on a similar argument to the convergence of the difference scheme.

To achieve these goals, we first prove the L^2-norm error estimate by a standard energy method. Then by taking inner products of different functions with the error system of linear equations, we obtain the error estimation in H^2-norm. More precisely, we take an inner product with a time difference quotient of the error function at the 1^{st} time level, and then we take another inner product for all the later levels. The techniques used during the proof include the technical energy method and detailed induction method. The embedding theorem in two dimensions makes the proof of the desired result in L^∞-norm coming true in the final step.

The rest of the paper is arranged as follows. In Section 2, some notations and basic lemmas are introduced to facilitate later numerical analysis. In Section 3, a three-level linearized compact difference scheme for the KT equation is derived at length. The main results including unique solvability, convergence and stability are proved in Section 4 followed by several numerical examples with different cases in Section 5. Some concluding remarks are drawn in Section 6.

2 Notations and auxiliary lemmas

We first introduce some notations and lemmas. Given a positive integer N, let $t_k=k\tau$, $0\le k\le N$, where $\tau=T/N$. Denote $\Omega_\tau=\{t_k|0\le k\le N\}$ and give a grid function $v=\{v^k|0\le k\le N\}$ on Ω_τ. Denote

$$\Delta_t v^k=\frac{1}{2\tau}(v^{k+1}-v^{k-1}), \qquad v^{\bar{k}}=\frac{1}{2}(v^{k+1}+v^{k-1}),$$
$$v^{k+\frac{1}{2}}=\frac{1}{2}(v^{k+1}+v^k), \qquad \delta_t v^{k+\frac{1}{2}}=\frac{1}{\tau}(v^{k+1}-v^k).$$

For two given positive integers M_1, M_2, let $h_1=L_1/M_1$, $h_2=L_2/M_2$, and denote $x_i=ih_1$, $y_j=jh_2$, $0\le i\le M_1$, $0\le j\le M_2$. Denote

$$\bar{\Omega}_h=\{(x_i,y_j)\,|\,0\le i\le M_1, 0\le j\le M_2\},\quad \Omega_h=\bar{\Omega}_h\cap\Omega,\quad \partial\Omega_h=\bar{\Omega}_h\cap\partial\Omega.$$

Define

$$\begin{aligned}&\bar{\omega}=\{(i,j)\,|\,0\le i\le M_1,\ 0\le j\le M_2\}, &&\omega=\{(i,j)\,|\,(x_i,y_j)\in\Omega_h\},\\ &\partial\omega=\bar{\omega}\setminus\omega, &&\mathcal{V}_h=\{v\,|\,v=\{v_{ij}\},\ 0\le i\le M_1,\ 0\le j\le M_2\},\end{aligned}$$

and

$$w_i=\begin{cases}1, & 1\le i\le M_1-1,\\ \dfrac{1}{2}, & i=0,M_1,\end{cases}\qquad \bar{w}_j=\begin{cases}1, & 1\le j\le M_2-1,\\ \dfrac{1}{2}, & j=0,M_2.\end{cases}$$

For any grid function $u\in\mathcal{V}_h$, we denote

$$\delta_x u_{i+\frac{1}{2},j}=\frac{1}{h_1}(u_{i+1,j}-u_{ij}),\quad \delta_y u_{i,j+\frac{1}{2}}=\frac{1}{h_2}(u_{i,j+1}-u_{ij}).$$

Introducing compact difference operators

$$\mathcal{A}_x u_{ij}=\begin{cases}\dfrac{5}{6}u_{0j}+\dfrac{1}{6}u_{1j}, & i=0,\\ \dfrac{1}{12}(u_{i-1,j}+10u_{ij}+u_{i+1,j}), & 1\le i\le M_1-1,\\ \dfrac{1}{6}u_{M_1-1,j}+\dfrac{5}{6}u_{M_1,j}, & i=M_1,\end{cases}$$

$$\delta_x^2 u_{ij}=\begin{cases}\dfrac{2}{h_1}\delta_x u_{\frac{1}{2},j}, & i=0,\\ \dfrac{1}{h_1}(\delta_x u_{i+\frac{1}{2},j}-\delta_x u_{i-\frac{1}{2},j}), & 1\le i\le M_1-1,\\ \dfrac{2}{h_1}(-\delta_x u_{M_1-\frac{1}{2},j}), & i=M_1.\end{cases}$$

Analogously, $\mathcal{A}_y u_{ij}$ and $\delta_y^2 u_{ij}$ can be defined. Denote

$$\Delta_h u_{ij}=\delta_x^2 u_{ij}+\delta_y^2 u_{ij},\quad \Lambda_h=\mathcal{A}_y\delta_x^2+\mathcal{A}_x\delta_y^2,\quad \mathcal{A}_h=\mathcal{A}_x\mathcal{A}_y.$$

Let u, $v\in\mathcal{V}_h$, the complex inner product of u and v is defined as

$$(u,v)=h_1h_2\sum_{i=0}^{M_1}\sum_{j=0}^{M_2}w_i\bar{w}_j u_{ij}\overline{v}_{ij}.$$

The corresponding norms or seminorms are defined as

$$\|u\|=\sqrt{(u,u)}, \qquad \|\delta_x u\|=\sqrt{h_1h_2\sum_{i=0}^{M_1-1}\sum_{j=0}^{M_2}\bar{w}_j|\delta_x u_{i+\frac{1}{2},j}|^2},$$

$$\|\nabla_h u\|=\sqrt{\|\delta_x u\|^2+\|\delta_y u\|^2}, \qquad \|\Delta_h u\|=\sqrt{h_1h_2\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1}(\Delta_h u_{ij})^2},$$

$$|u|_1=\|\nabla_h u\|, \qquad \|u\|_\infty=\max_{0\le i\le M_1,\ 0\le j\le M_2}|u_{ij}|.$$

Similarly, $\|\delta_y u\|$ can be defined.

The following lemmas play important roles in the later numerical analysis.

Lemma 2.1. *For any grid function $u\in\mathcal{V}_h$, it holds that*

$$(\mathcal{A}_x u,u)=\|u\|^2-\frac{h_1^2}{12}\|\delta_x u\|^2, \quad (\mathcal{A}_y u,u)=\|u\|^2-\frac{h_2^2}{12}\|\delta_y u\|^2,$$

and

$$\frac{2}{3}\|u\|^2\le(\mathcal{A}_x u,u)\le\|u\|^2, \quad \frac{2}{3}\|u\|^2\le(\mathcal{A}_y u,u)\le\|u\|^2.$$

Proof. From the definition of $\mathcal{A}_x$ and δ_x^2, we know $\mathcal{A}_x u_{ij}=(1+h_1^2/12\delta_x^2)u_{ij}$. According to the definition of the inner product, we have

$$\begin{aligned}
(\mathcal{A}_x u,u)&=h_1h_2\sum_{i=0}^{M_1}\sum_{j=0}^{M_2}\omega_i\bar{\omega}_j(\mathcal{A}_x u_{ij})\overline{u}_{ij}\\
&=h_1h_2\sum_{i=0}^{M_1}\sum_{j=0}^{M_2}\omega_i\bar{\omega}_j\Big(1+\frac{h_1^2}{12}\delta_x^2\Big)u_{ij}\overline{u}_{ij}\\
&=h_1h_2\sum_{i=0}^{M_1}\sum_{j=0}^{M_2}\omega_i\bar{\omega}_j u_{ij}\overline{u}_{ij}+h_1h_2\sum_{i=0}^{M_1}\sum_{j=0}^{M_2}\omega_i\bar{\omega}_j\frac{h_1^2}{12}(\delta_x^2 u_{ij})\overline{u}_{ij}\\
&=\|u\|^2+\frac{h_1^2}{12}h_2\sum_{j=0}^{M_2}\bar{\omega}_j\left(-h_1\sum_{i=0}^{M_1-1}|\delta_x u_{i+\frac{1}{2},j}|^2\right)\\
&=\|u\|^2-\frac{h_1^2}{12}\cdot h_1h_2\sum_{i=0}^{M_1-1}\sum_{j=0}^{M_2}\bar{\omega}_j|\delta_x u_{i+\frac{1}{2},j}|^2\\
&=\|u\|^2-\frac{h_1^2}{12}\|\delta_x u\|^2.
\end{aligned}$$

Similarly,

$$(\mathcal{A}_y u,u)=\|u\|^2-\frac{h_2^2}{12}\|\delta_y u\|^2.$$

It is easy to know that

$$(\mathcal{A}_x u,u)\leq\|u\|^2,\quad (\mathcal{A}_y u,u)\leq\|u\|^2.$$

In addition,

$$\begin{aligned}\|\delta_x u\|^2=&h_1h_2\sum_{i=0}^{M_1-1}\sum_{j=0}^{M_2}\bar{\omega}_j|\delta_x u_{i+\frac{1}{2},j}|^2\\=&h_1h_2\sum_{i=0}^{M_1-1}\sum_{j=0}^{M_2}\bar{\omega}_j\left|\frac{u_{i+1,j}-u_{ij}}{h_1}\right|^2\\\leq&2h_1h_2\sum_{i=0}^{M_1-1}\sum_{j=0}^{M_2}\bar{\omega}_j\frac{1}{h_1^2}(|u_{i+1,j}|^2+|u_{ij}|^2)\\\leq&\frac{4}{h_1^2}h_1h_2\sum_{i=0}^{M_1}\sum_{j=0}^{M_2}\omega_i\bar{\omega}_j|u_{ij}|^2\leq\frac{4}{h_1^2}\|u\|^2.\end{aligned}$$

Thus,

$$(\mathcal{A}_x u,u)=\|u\|^2-\frac{h_1^2}{12}\|\delta_x u\|^2\geq\|u\|^2-\frac{h_1^2}{12}\frac{4}{h_1^2}\|u\|^2=\frac{2}{3}\|u\|^2.$$

In the same manner, the inequality for the y-direction holds. This completes this proof. □

The proof of the above lemma implies the following lemmas.

Lemma 2.2 ([10])**.** *For any grid function $u\in\mathcal{V}_h$, it holds that*

$$(\delta_x^2 u,u)=-\|\delta_x u\|^2,\quad (\delta_y^2 u,u)=-\|\delta_y u\|^2,\quad \frac{1}{3}\|u\|^2\leq(\mathcal{A}_h u,u)\leq\|u\|^2.$$

Lemma 2.3. *For any grid functions $u,v\in\mathcal{V}_h$, it holds that*

$$\mathrm{Re}(\mathcal{A}_h u,v)=\mathrm{Re}(\mathcal{A}_h v,u).$$

Proof. The results can be proved similar to Lemma 2.1. □

Lemma 2.4. *For any grid function $u\in\mathcal{V}_h$, it holds that*

$$
\begin{aligned}
&-(\mathcal{A}_h u,\Delta_h u)\geq \frac{1}{3}|u|_1^2,\\
&(\Lambda_h u,\Delta_h u)=\|\Delta_h u\|^2-\frac{h_1^2+h_2^2}{12}(\|\delta_x^2\delta_y u\|^2+\|\delta_x\delta_y^2 u\|^2),\\
&\frac{2}{3}\|\Delta_h u\|^2\leq(\Lambda_h u,\Delta_h u)\leq\|\Delta_h u\|^2.
\end{aligned}
$$

Proof. This result can be obtained directly as that in [3]. □

Lemma 2.5 ([3])**.** *Assume that the grid function $w\in\mathcal{V}_h$ is bounded in H^1-seminorm and L^∞-norm, namely there exists a constant $\hat{c}$ such that $|w|_1\leq\hat{c}$, $\|w\|_\infty\leq\hat{c}$, then for any functions $u,v\in\mathcal{V}_h$, it holds that*

$$
|(\mathcal{A}_h(wu),\Delta_h v)|\leq\sqrt{2}\hat{c}(\|u\|_\infty+|u|_1)|v|_1.
$$

Lemma 2.6 ([3])**.** *For any grid function $v\in\mathcal{V}_h$, there exists a positive constant $\hat{c}$ such that*

$$
|v|_1\leq\hat{c}\|\Delta_h v\|.
$$

Lemma 2.7 ([15])**.** *For any grid function $u\in\mathcal{V}_h$ and arbitrary $\varepsilon>0$, it holds that*

$$
\|u\|_\infty\leq\varepsilon\|\Delta_h u\|+\sqrt{3}\left[\frac{1}{\varepsilon}+\frac{1}{2}\left(\frac{1}{L_1}+\frac{1}{L_2}\right)\right]\cdot\|u\|\leq\hat{c}(\|\Delta_h u\|+\|u\|).
$$

Lemma 2.8 ([15])**.** *Suppose $\{E^k\}_{k=0}^\infty$ is a nonnegative sequence satisfying*

$$
E^{k+1}\leq(1+c\tau)E^k+\tau g,\quad k=0,1,2,\cdots.
$$

Then it holds

$$
E^k\leq\exp(ck\tau)\left(E^0+\frac{g}{c}\right),\quad k=1,2,\cdots,
$$

where c and g are nonnegative constants.

Throughout the whole paper, we assume that the nonlinear KT equation (1.1) allows a unique solution and the solution u is bounded by

$$
\|u\|_{L^\infty([0,T],H^6(\Omega))}+\|u_t\|_{L^\infty([0,T],H^3(\Omega))}+\|u_{tt}\|_{L^\infty([0,T],H^2(\Omega))}+\|u\|_{L^\infty([0,T],H^1(\Omega))}.
$$

However, it is worth noting that several important work by the low-order integrators [1, 8] with the harmonic analysis technique have been proposed, which could weaken this regularity condition.

3 The derivation of the difference scheme

Defining the mesh functions

$$U_{ij}^k = u(x_i,y_j,t_k), \quad (i,j)\in\bar{\omega}, \quad 0\le k\le N.$$

Considering (1.1a) at the point (x_i,y_j,t_k), we have

$$\begin{aligned} u_t(x_i,y_j,t_k) =& (1+\mathrm{i}c_1)\Delta u(x_i,y_j,t_k)+\gamma u(x_i,y_j,t_k) \\ &-(1+\mathrm{i}c_2)|u(x_i,y_j,t_k)|^2 u(x_i,y_j,t_k), \quad (i,j)\in\bar{\omega}, \quad 1\le k\le N-1. \end{aligned} \tag{3.1}$$

Taking the operator $\mathcal{A}_h$ on both sides of (3.1) and using the Taylor expansion, we have

$$\begin{aligned} \mathcal{A}_h\Delta_t U_{ij}^k =& (1+\mathrm{i}c_1)\Lambda_h U_{ij}^{\bar{k}}+\gamma\mathcal{A}_h U_{ij}^{\bar{k}}-(1+\mathrm{i}c_2)\mathcal{A}_h(|U_{ij}^k|^2 U_{ij}^{\bar{k}}) \\ &+R_{ij}^k, (i,j)\in\bar{\omega}, \quad 1\le k\le N-1, \end{aligned} \tag{3.2}$$

where

$$|R_{ij}^k|\le C_R(\tau^2+h_1^4+h_2^4),$$

and C_R is independent of τ, h_1 and h_2. Moreover, we have

$$|\Delta_t R_{ij}^k|\le C_R(\tau^2+h_1^4+h_2^4).$$

For the first level in time, a linearized Crank-Nicolson scheme is utilized. Considering (1.1a) at the point $(x_i,y_j,t_{\frac{1}{2}})$ and taking the operator $\mathcal{A}_h$ on both sides, we have

$$\mathcal{A}_h\delta_t U_{ij}^{\frac{1}{2}} = (1+\mathrm{i}c_1)\Lambda_h U_{ij}^{\frac{1}{2}}+\gamma\mathcal{A}_h U_{ij}^{\frac{1}{2}}-(1+\mathrm{i}c_2)\mathcal{A}_h(|\hat{u}_{ij}^0|^2 U_{ij}^{\frac{1}{2}})+R_{ij}^0, \quad (i,j)\in\bar{\omega}, \tag{3.3}$$

where

$$\hat{u}_{ij}^0 = U_{ij}^0+\frac{\tau}{2}u_t(x_i,y_j,0)$$

and

$$|R_{ij}^0|\le C_R(\tau^2+h_1^4+h_2^4).$$

Noticing the initial condition (1.1b), we have

$$U_{ij}^0 = \varphi(x_i,y_j), \quad (i,j)\in\bar{\omega}. \tag{3.4}$$

Omitting the small terms R_{ij}^k in (3.2) and R_{ij}^0 in (3.3), and replacing the exact solution U_{ij}^k with the numerical solution u_{ij}^k, the linearized compact difference scheme reads

$$\begin{cases}\mathcal{A}_h\delta_t u_{ij}^{\frac{1}{2}}=(1+\mathrm{i}c_1)\Lambda_h u_{ij}^{\frac{1}{2}}+\gamma\mathcal{A}_h u_{ij}^{\frac{1}{2}}\\ \qquad\qquad -(1+\mathrm{i}c_2)\mathcal{A}_h(|\hat{u}_{ij}^0|^2 u_{ij}^{\frac{1}{2}}),\quad (i,j)\in\bar{\omega}, & (3.5\text{a})\\ \mathcal{A}_h\Delta_t u_{ij}^k=(1+\mathrm{i}c_1)\Lambda_h u_{ij}^{\bar{k}}+\gamma\mathcal{A}_h u_{ij}^{\bar{k}}\\ \qquad\qquad -(1+\mathrm{i}c_2)\mathcal{A}_h(|u_{ij}^k|^2 u_{ij}^{\bar{k}}),\quad (i,j)\in\bar{\omega},\quad 1\le k\le N-1, & (3.5\text{b})\\ u_{ij}^0=\varphi(x_i,y_j),\qquad (i,j)\in\bar{\omega}. & (3.5\text{c})\end{cases}$$

4 The numerical analysis

Denote

$$u^k=(u_{0,0}^k,u_{1,0}^k,\cdots,u_{M_1,0}^k,u_{0,1}^k,u_{1,1}^k,\cdots,u_{M_1,1}^k,\cdots,u_{0,M_2}^k,u_{1,M_2}^k,\cdots,u_{M_1,M_2}^k)^T.$$

4.1 Unique solvability

Theorem 4.1. *Denote*

$$\hat{c}_0=\frac{|\gamma|}{2}+\frac{1}{2}\sqrt{1+c_2^2}\cdot\||\hat{u}^0|^2\|_\infty,$$

when $\tau\le\frac{1}{3\hat{c}_0}$*, the compact difference scheme* (3.5a)–(3.5c) *is uniquely solvable.*

Proof. According to (3.5c), it is easy to know that u^0 is uniquely determined. Considering the homogeneous system of Eq. (3.5a), we have

$$\frac{1}{\tau}\mathcal{A}_h u_{ij}^1=\frac{1}{2}(1+\mathrm{i}c_1)\Lambda_h u_{ij}^1+\frac{\gamma}{2}\mathcal{A}_h u_{ij}^1-\frac{1}{2}(1+\mathrm{i}c_2)\mathcal{A}_h|\hat{u}_{ij}^0|u_{ij}^1,\quad (i,j)\in\bar{\omega}. \tag{4.1}$$

Taking the inner product of (4.1) with u^1 yields

$$\frac{1}{\tau}(\mathcal{A}_h u^1,u^1)=\frac{1}{2}(1+\mathrm{i}c_1)(\Lambda_h u^1,u^1)+\frac{\gamma}{2}(\mathcal{A}_h u^1,u^1)-\frac{1}{2}(1+\mathrm{i}c_2)(\mathcal{A}_h(|\hat{u}^0|u^1),u^1). \tag{4.2}$$

Noticing Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}(\Lambda_h u^1,u^1)&=((\mathcal{A}_x\delta_y^2+\mathcal{A}_y\delta_x^2)u^1,u^1)\\ &=((\mathcal{A}_x\delta_y^2)u^1,u^1)+((\mathcal{A}_y\delta_x^2)u^1,u^1)\\ &=-((\mathcal{A}_x\delta_y)u^1,\delta_y u^1)-((\mathcal{A}_y\delta_x)u^1,\delta_x u^1)\\ &\le-\frac{2}{3}(\|\delta_y u^1\|^2+\|\delta_x u^1\|^2)=-\frac{2}{3}|u^1|_1^2.\end{aligned} \tag{4.3}$$

Taking the real part on both sides of (4.2), we have

$$\begin{aligned}\frac{1}{3\tau}\|u^1\|^2 &\le -\frac{1}{3}|u^1|_1^2+\frac{|\gamma|}{2}\|u^1\|^2+\frac{1}{2}\mathrm{Re}\{-(1+\mathrm{i}c_2)(\mathcal{A}_h(|\hat{u}^0|^2u^1),u^1)\}\\ &\le -\frac{1}{3}|u^1|_1^2+\frac{|\gamma|}{2}\|u^1\|^2+\frac{1}{2}\sqrt{1+c_2^2}\cdot\||\hat{u}^0|^2\|_\infty\cdot\|u^1\|^2\le \hat{c}_0\|u^1\|^2. \qquad (4.4)\end{aligned}$$

In other words,

$$(1-3\hat{c}_0\tau)\|u^1\|^2\le 0.$$

Thus, when $\tau<\frac{1}{3\hat{c}_0}$, $\|u^1\|=0$, which implies that $u_{ij}^1=0$ for $(i,j)\in\bar{\omega}$. Therefore, (3.5a) and (3.5c) uniquely determine u^1.

Suppose u^k and u^{k-1} have been determined. Then, considering the homogeneous system of Eq. (3.5b), we have

$$\begin{aligned}\frac{1}{2\tau}\mathcal{A}_h u_{ij}^{k+1}=&\frac{1}{2}(1+\mathrm{i}c_1)\Lambda_h u_{ij}^{k+1}+\frac{\gamma}{2}\mathcal{A}_h u_{ij}^{k+1}\\ &-\frac{1}{2}(1+\mathrm{i}c_2)\mathcal{A}_h(|u_{ij}^k|^2u_{ij}^{k+1}),\quad (i,j)\in\bar{\omega}. \qquad (4.5)\end{aligned}$$

Taking the inner product of (4.5) with u^{k+1}, we have

$$\begin{aligned}\frac{1}{2\tau}(\mathcal{A}_h u^{k+1},u^{k+1})=&\frac{1}{2}(1+\mathrm{i}c_1)(\Lambda_h u^{k+1},u^{k+1})+\frac{\gamma}{2}(\mathcal{A}_h u^{k+1},u^{k+1})\\ &-\frac{1}{2}(1+\mathrm{i}c_2)(\mathcal{A}_h|u^k|^2u^{k+1},u^{k+1}). \qquad (4.6)\end{aligned}$$

Taking the real part on both sides of (4.6), and then utilizing

$$\begin{aligned}&(\mathcal{A}_h u^{k+1},u^{k+1})\ge\frac{1}{3}\|u^{k+1}\|^2,\\ &(\Lambda_h u^{k+1},u^{k+1})\le-\frac{2}{3}|u^{k+1}|_1^2,\\ &(\mathcal{A}_h u^{k+1},u^{k+1})\le\|u^{k+1}\|^2,\end{aligned}$$

we have

$$\begin{aligned}\frac{1}{6\tau}\|u^{k+1}\|^2&\le-\frac{1}{3}|u^{k+1}|_1^2+\frac{|\gamma|}{2}\|u^{k+1}\|^2-\frac{1}{2}\mathrm{Re}\{(1+\mathrm{i}c_2)(\mathcal{A}_h|u^k|^2u^{k+1},u^{k+1})\}\\ &\le\frac{|\gamma|}{2}\|u^{k+1}\|^2,\end{aligned}$$

by noticing that

$$\begin{aligned}&-\frac{1}{2}\mathrm{Re}\{(1+\mathrm{i}c_2)(\mathcal{A}_h|u^k|^2u^{k+1},u^{k+1})\}\\ =&-\frac{1}{2}(\mathcal{A}_h|u^k|^2u^{k+1},u^{k+1})\le 0.\end{aligned}$$

Namely,

$$(1-3|\gamma|\tau)\|u^{k+1}\|^2\le 0.$$

Thus, if $\gamma=0$, the above inequality implies that $u_{ij}^{k+1}=0$ naturally. If $\gamma\neq 0$, we have $\|u^{k+1}\|=0$ when $\tau<\frac{1}{3|\gamma|}$. Therefore, u^{k+1} has been uniquely determined. By induction, this completes the proof. □

4.2 Convergence

Denote

$$C_u=\max_{(x,y)\in\Omega,\ 0\le t\le T}|u(x,y,t)|,$$

and

$$\hat{c}_1=|\gamma|+\frac{1}{2}+\hat{c}_0\sqrt{1+c_2^2},\qquad \hat{c}_2=\sqrt{1+c_2^2}(3C_u^2+3C_u+2),$$

$$\hat{c}_3=3(1+2|\gamma|+2\hat{c}_2),\qquad \hat{c}_4=\sqrt{3\left(6+\frac{1}{\hat{c}_3}\right)T}\exp(2\hat{c}_3T)\sqrt{L_1L_2}C_R,$$

$$\hat{c}_5=\frac{81}{2}\left[\gamma^2+(1+c_2^2)\||\widehat{u}^0|^2\|_\infty^2\right]+27,\qquad \hat{c}_6=18+2TC_R^2,$$

$$\hat{c}_7=6\hat{c}_6\exp(6\hat{c}_5T),\qquad \hat{c}_8=\hat{c}(\hat{c}_4+\hat{c}_7).$$

We have the following convergence result.

Theorem 4.2. *Suppose that* $\{U_{ij}^k|(i,j)\in\bar{\omega},\ 0\le k\le N\}$ *is the solution of* (1.1a)–(1.1c) *and* $\{u_{ij}^k|(i,j)\in\bar{\omega},\ 0\le k\le N\}$ *is the solution of the linearized compact difference scheme* (3.5a)–(3.5c). *Denote*

$$e_{ij}^k=U_{ij}^k-u_{ij}^k,\quad (i,j)\in\bar{\omega},\quad 0\le k\le N.$$

When $\tau^2+h_1^4+h_2^4\le 1/\hat{c}_8$ *and* $\tau\le 1$, *then we have*

$$\|e^k\|_\infty\le\hat{c}_8(\tau^2+h_1^4+h_2^4),\quad 0\le k\le N. \tag{4.7}$$

Proof. Subtracting (3.5a)–(3.5c) from (3.2)–(3.4), we have the following error system

$$\begin{cases}\mathcal{A}_h\delta_te_{ij}^{\frac{1}{2}}=(1+\mathrm{i}c_1)\Lambda_he_{ij}^{\frac{1}{2}}+\gamma\mathcal{A}_he_{ij}^{\frac{1}{2}}-(1+\mathrm{i}c_2)\mathcal{A}_h(|\hat{u}_{ij}^0|^2e_{ij}^{\frac{1}{2}})\\ \qquad +R_{ij}^0,\quad (i,j)\in\bar{\omega}, & (4.8\text{a})\\ \mathcal{A}_h\Delta_te_{ij}^k=(1+\mathrm{i}c_1)\Lambda_he_{ij}^{\bar{k}}+\gamma\mathcal{A}_he_{ij}^{\bar{k}}-(1+\mathrm{i}c_2)\mathcal{A}_h(|U_{ij}^k|^2U_{ij}^{\bar{k}}-|u_{ij}^k|^2u_{ij}^{\bar{k}})\\ \qquad +R_{ij}^k,\quad (i,j)\in\bar{\omega},\quad 1\le k\le N-1, & (4.8\text{b})\\ e_{ij}^0=0,\qquad (i,j)\in\bar{\omega}. & (4.8\text{c})\end{cases}$$

The proof is divided into three steps.

Step 1: $\|\cdot\|$ error estimate. By (4.8c), we have $\|e^0\|=0$. Taking the inner product of (4.8a) with $e^{\frac{1}{2}}$, we have

$$\begin{aligned}(\mathcal{A}_h\delta_t e^{\frac{1}{2}},e^{\frac{1}{2}})=&(1+\mathrm{i}c_1)(\Lambda_h e^{\frac{1}{2}},e^{\frac{1}{2}})+\gamma(\mathcal{A}_h e^{\frac{1}{2}},e^{\frac{1}{2}})\\&-(1+\mathrm{i}c_2)(\mathcal{A}_h(|\hat{u}^0|^2e^{\frac{1}{2}}),e^{\frac{1}{2}})+(R^0,e^{\frac{1}{2}}).\end{aligned}\tag{4.9}$$

Noticing that

$$\begin{cases}(\mathcal{A}_h\delta_t e^{\frac{1}{2}},e^{\frac{1}{2}})=\dfrac{1}{2\tau}(\mathcal{A}_h e^1,e^1),\\(\Lambda_h e^{\frac{1}{2}},e^{\frac{1}{2}})\le-\dfrac{2}{3}|e^{\frac{1}{2}}|_1^2,\\(\mathcal{A}_h e^{\frac{1}{2}},e^{\frac{1}{2}})\le\|e^{\frac{1}{2}}\|^2,\\(\mathcal{A}_h(|\hat{u}^0|^2e^{\frac{1}{2}}),e^{\frac{1}{2}})\le\||\hat{u}^0|^2\|_\infty\cdot\|e^{\frac{1}{2}}\|^2\le\hat{c}_0\|e^{\frac{1}{2}}\|^2,\\|(R^0,e^{\frac{1}{2}})|\le\|R^0\|\cdot\|e^{\frac{1}{2}}\|\le\dfrac{1}{2}\|R^0\|^2+\dfrac{1}{2}\|e^{\frac{1}{2}}\|^2,\end{cases}$$

and taking the real part of (4.9) yields

$$\begin{aligned}\frac{1}{2\tau}(\mathcal{A}_h e^1,e^1)\le&-\frac{2}{3}|e^{\frac{1}{2}}|_1^2+|\gamma|\cdot|\|e^{\frac{1}{2}}\|^2+\mathrm{Re}\Big\{-(1+\mathrm{i}c_2)(\mathcal{A}_h(|\hat{u}^0|^2e^{\frac{1}{2}}),e^{\frac{1}{2}})\Big\}+\mathrm{Re}(R^0,e^{\frac{1}{2}})\\\le&|\gamma|\cdot\|e^{\frac{1}{2}}\|^2+\hat{c}_0\sqrt{1+c_2^2}\|e^{\frac{1}{2}}\|^2+\frac{1}{2}\|R^0\|^2+\frac{1}{2}\|e^{\frac{1}{2}}\|^2\\\le&\hat{c}_1\|e^{\frac{1}{2}}\|^2+\frac{1}{2}\|R^0\|^2.\end{aligned}$$

Noticing that

$$(\mathcal{A}_h e^1,e^1)\ge\frac{1}{3}\|e^1\|^2,$$

we have

$$\|e^1\|^2\le6\hat{c}_1\tau\|e^{\frac{1}{2}}\|^2+3\tau\|R^0\|^2\le\frac{3}{2}\hat{c}_1\tau\|e^1\|^2+3\tau\|R^0\|^2,$$

namely,

$$\Big(1-\frac{3}{2}\hat{c}_1\tau\Big)\|e^1\|^2\le3\tau\|R^0\|^2.$$

When $\frac{3}{2}\hat{c}_1\tau\le\frac{1}{2}$, we have

$$\|e^1\|^2\le6\tau\|R^0\|^2\le6\tau L_1L_2C_R^2(\tau^2+h_1^4+h_2^4)^2.\tag{4.11}$$

Next, we assume that (4.7) holds for $1\le k\le l\le N-1$. Then we need to prove that it holds for $k=l+1$. According to inductive hypothesis, when $\tau^2+h_1^4+h_2^4\le 1/\hat{c}_8$, we have

$$\|u^k\|_\infty=\|U^k-(U^k-u^k)\|_\infty\le\|U^k\|_\infty+\|e^k\|_\infty\le C_u+1,\quad 1\le k\le l. \tag{4.12}$$

Therefore, we have

$$\||u^k|^2\|_\infty\le(C_u+1)^2,\quad \||U^ku^k|\|_\infty\le C_u(C_u+1),\quad \||U^k|^2\|_\infty\le C_u^2,\quad 1\le k\le l.$$

Taking the inner product of (4.8b) with $e^{\bar{k}}$, we have

$$\begin{aligned}(\mathcal{A}_h\Delta_te^k,e^{\bar{k}})=&(1+\mathrm{i}c_1)(\Lambda_he^{\bar{k}},e^{\bar{k}})+\gamma(\mathcal{A}_he^{\bar{k}},e^{\bar{k}})-(1+\mathrm{i}c_2)(\mathcal{A}_h(|U^k|^2U^{\bar{k}}-|u^k|^2u^{\bar{k}}),e^{\bar{k}})\\&+(R^k,e^{\bar{k}}),\quad 1\le k\le N-1. \end{aligned}\tag{4.13}$$

By Lemmas 2.3 and 2.2, we have

$$\mathrm{Re}(\mathcal{A}_h\Delta_te^k,e^{\bar{k}})=\frac{1}{4\tau}[(\mathcal{A}_he^{k+1},e^{k+1})-(\mathcal{A}_he^{k-1},e^{k-1})], \tag{4.14a}$$

$$(\Lambda_he^{\bar{k}},e^{\bar{k}})\le-\frac{2}{3}|e^{\bar{k}}|_1^2,\quad (\mathcal{A}_he^{\bar{k}},e^{\bar{k}})\le\|e^{\bar{k}}\|^2. \tag{4.14b}$$

Noticing that

$$\begin{aligned}|U^k|^2U^{\bar{k}}-|u^k|^2u^{\bar{k}}&=(|U^k|^2U^{\bar{k}}-|U^k|^2u^{\bar{k}})+(|U^k|^2u^{\bar{k}}-|u^k|^2u^{\bar{k}})\\&=|U^k|^2e^{\bar{k}}+(U^k\overline{U^k}-u^k\overline{u^k})u^{\bar{k}}\\&=|U^k|^2e^{\bar{k}}+(U^k\overline{U^k}-U^k\overline{u^k}+U^k\overline{u^k}-u^k\overline{u^k})u^{\bar{k}}\\&=|U^k|^2e^{\bar{k}}+(U^k\overline{e^k}+e^k\overline{u^k})u^{\bar{k}},\end{aligned}$$

we have

$$\begin{aligned}&\big(\mathcal{A}_h(|U^k|^2U^{\bar{k}}-|u^k|^2u^{\bar{k}}),e^{\bar{k}}\big)\\\le&\big|(\mathcal{A}_h(|U^k|^2e^{\bar{k}}),e^{\bar{k}})\big|+\big|(\mathcal{A}_h(U^k\overline{e^k}u^{\bar{k}}),e^{\bar{k}})\big|+\big|(\mathcal{A}_h(e^k\overline{u^k}u^{\bar{k}}),e^{\bar{k}})\big|\\\le&\|\mathcal{A}_h(|U^k|^2e^{\bar{k}})\|\cdot\|e^{\bar{k}}\|+\|\mathcal{A}_h(U^k\overline{e^k}u^{\bar{k}})\|\cdot\|e^{\bar{k}}\|+\|\mathcal{A}_h(e^k\overline{u^k}u^{\bar{k}})\|\cdot\|e^{\bar{k}}\|\\\le&C_u^2\|e^{\bar{k}}\|\cdot\|e^{\bar{k}}\|+C_u(C_u+1)\|e^k\|\cdot\|e^{\bar{k}}\|+(C_u+1)^2\|e^k\|\cdot\|e^{\bar{k}}\|\\\le&(C_u^2+C_u(C_u+1)+(C_u+1)^2)(\|e^{\bar{k}}\|^2+\|e^k\|^2). \end{aligned}\tag{4.15}$$

Taking the real part of (4.13) and inserting (4.14a)–(4.15), we have

$$\begin{aligned}&\frac{1}{4\tau}\big[(\mathcal{A}_he^{k+1},e^{k+1})-(\mathcal{A}_he^{k-1},e^{k-1})\big]\\\le&-\frac{2}{3}|e^{\bar{k}}|_1^2+|\gamma|\cdot\|e^{\bar{k}}\|^2-\mathrm{Re}\Big\{(1+\mathrm{i}c_2)(\mathcal{A}_h(|U^k|^2U^{\bar{k}}-|u^k|^2u^{\bar{k}}),e^{\bar{k}})\Big\}+\mathrm{Re}(R^k,e^{\bar{k}})\\\le&|\gamma|\cdot\|e^{\bar{k}}\|^2+\sqrt{1+c_2^2}(3C_u^2+3C_u+2)(\|e^{\bar{k}}\|^2+\|e^k\|^2)+\|R^k\|\cdot\|e^{\bar{k}}\|,\quad 1\le k\le l.\end{aligned}$$

Denote

$$F^{k+1}=(\mathcal{A}_h e^{k+1},e^{k+1})+(\mathcal{A}_h e^k,e^k).$$

According to Lemma 2.2, we have

$$\begin{aligned}
F^{k+1}-F^k &\le 4\tau|\gamma|\cdot\|e^{\bar{k}}\|^2+4\tau\hat{c}_2(\|e^{\bar{k}}\|^2+\|e^k\|^2)+4\tau\|R^k\|\cdot\|e^{\bar{k}}\| \\
&\le 2\tau(1+2|\gamma|+2\hat{c}_2)(\|e^{\bar{k}}\|^2+\|e^k\|^2)+2\tau\|R^k\|^2 \\
&\le 3\tau(1+2|\gamma|+2\hat{c}_2)[(\mathcal{A}_h e^{k+1},e^{k+1})+2(\mathcal{A}_h e^k,e^k)+(\mathcal{A}_h e^{k-1},e^{k-1})]+2\tau\|R^k\|^2 \\
&\le \hat{c}_3\tau(F^{k+1}+F^k)+2\tau\|R^k\|^2, \quad 1\le k\le l.
\end{aligned}$$

When $\hat{c}_3\tau\le\frac{1}{2}$, we have

$$\begin{aligned}
F^{k+1} &\le (1+4\hat{c}_3\tau)F^k+4\tau\|R^k\|^2 \\
&\le (1+4\hat{c}_3\tau)F^k+4\tau L_1L_2C_R^2(\tau^2+h_1^4+h_2^4)^2, \quad 1\le k\le l.
\end{aligned}$$

Using the Gronwall inequality, when $\tau\le 1$, we have

$$\begin{aligned}
F^{k+1} &\le \exp(4\hat{c}_3k\tau)\Big[F^1+\frac{1}{\hat{c}_3}L_1L_2C_R^2(\tau^2+h_1^4+h_2^4)^2\Big] \\
&\le \exp(4\hat{c}_3T)\Big[(\mathcal{A}_h e^1,e^1)+\frac{1}{\hat{c}_3}L_1L_2C_R^2(\tau^2+h_1^4+h_2^4)^2\Big] \\
&\le \Big(6+\frac{1}{\hat{c}_3}\Big)T\exp(4\hat{c}_3T)L_1L_2C_R^2(\tau^2+h_1^4+h_2^4)^2, \quad 1\le k\le l.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\|e^{k+1}\|^2 &\le 3\Big(6+\frac{1}{\hat{c}_3}\Big)T\exp(4\hat{c}_3T)L_1L_2C_R^2(\tau^2+h_1^4+h_2^4)^2 \\
&:= \hat{c}_4^2(\tau^2+h_1^4+h_2^4)^2, \quad 1\le k\le l. \qquad (4.16)
\end{aligned}$$

Step 2: $\|\Delta_h e^k\|$ error estimate. Taking the inner product of (4.8a) with $-\delta_t\Delta_h e^{\frac{1}{2}}$ at the 1st time level, we have

$$\begin{aligned}
(\mathcal{A}_h\delta_t e^{\frac{1}{2}},-\delta_t\Delta_h e^{\frac{1}{2}}) =& (1+\mathrm{i}c_1)(\Lambda_h e^{\frac{1}{2}},-\delta_t\Delta_h e^{\frac{1}{2}})+\gamma(\mathcal{A}_h e^{\frac{1}{2}},-\delta_t\Delta_h e^{\frac{1}{2}}) \\
&-(1+\mathrm{i}c_2)(\mathcal{A}_h(|\hat{u}^0|^2e^{\frac{1}{2}}),-\delta_t\Delta_h e^{\frac{1}{2}})+(R^0,-\delta_t\Delta_h e^{\frac{1}{2}}). \qquad (4.17)
\end{aligned}$$

Using Lemma 2.4, we have

$$\begin{cases}(\mathcal{A}_h\delta_t e^{\frac{1}{2}},-\delta_t\Delta_h e^{\frac{1}{2}})\geq\frac{1}{3}|\delta_t e^{\frac{1}{2}}|_1^2=\frac{1}{3\tau^2}|e^1|_1^2,\\ (\Lambda_h e^{\frac{1}{2}},-\delta_t\Delta_h e^{\frac{1}{2}})=-\frac{1}{2\tau}(\Lambda_h e^1,\Delta_h e^1),\\ \mathrm{Re}(\mathcal{A}_h e^{\frac{1}{2}},-\delta_t\Delta_h e^{\frac{1}{2}})=-\frac{1}{2\tau}\mathrm{Re}(\mathcal{A}_h e^1,\Delta_h e^1)\leq\frac{1}{2\tau}\|\mathcal{A}_h e^1\|\cdot\|\Delta_h e^1\|\leq\frac{1}{2\tau}\|e^1\|\cdot\|\Delta_h e^1\|,\\ (\mathcal{A}_h(|\hat{u}^0|^2 e^{\frac{1}{2}}),\delta_t\Delta_h e^{\frac{1}{2}})\leq\frac{1}{2\tau}\||\hat{u}^0|^2\|_\infty\cdot\|\mathcal{A}_h e^1\|\cdot\|\Delta_h e^1\|\leq\frac{1}{2\tau}\||\hat{u}^0|^2\|_\infty\|e^1\|\cdot\|\Delta_h e^1\|,\\ \mathrm{Re}(R^0,-\delta_t\Delta_h e^{\frac{1}{2}})=-\frac{1}{\tau}\mathrm{Re}(R^0,\Delta_h e^1)\leq\frac{1}{\tau}\|R^0\|\cdot\|\Delta_h e^1\|.\end{cases}$$

Taking the real part of (4.17), and combining Lemma 2.4 with the above inequalities, we have

$$\begin{aligned}\frac{1}{3\tau^2}|e^1|_1^2\leq&-\frac{1}{2\tau}\mathrm{Re}(\Lambda_h e^1,-\Delta_h e^1)+\frac{|\gamma|}{2\tau}\|e^1\|\cdot\|\Delta_h e^1\|\\&+\mathrm{Re}\Big\{(1+\mathrm{i}c_2)(\mathcal{A}_h(|\hat{u}^0|^2 e^{\frac{1}{2}}),\delta_t\Delta_h e^{\frac{1}{2}})\Big\}+\mathrm{Re}(R^0,-\delta_t\Delta_h e^{\frac{1}{2}})\\ \leq&-\frac{1}{3\tau}\|\Delta_h e^1\|^2+\frac{|\gamma|}{2\tau}\|e^1\|\cdot\|\Delta_h e^1\|\\&+\frac{1}{2\tau}\sqrt{1+c_2^2}\||\hat{u}^0|^2\|_\infty\cdot\|e^1\|\cdot\|\Delta_h e^1\|+\frac{1}{\tau}\|R^0\|\cdot\|\Delta_h e^1\|.\end{aligned}\tag{4.18}$$

Rearranging (4.18) and when $\tau\leq 1$, we have

$$\begin{aligned}\frac{1}{3}\|\Delta_h e^1\|^2\leq&\frac{1}{18}\|\Delta_h e^1\|^2+\frac{9|\gamma|^2}{8}\|e^1\|^2+\frac{1}{18}\|\Delta_h e^1\|^2\\&+\frac{9}{8}(1+c_2^2)\||\widehat{u}^0|^2\|_\infty^2\cdot\|e^1\|^2+\frac{1}{18}\|\Delta_h e^1\|^2+\frac{9}{2}\|R^0\|^2.\end{aligned}$$

Thus, we have

$$\|\Delta_h e^1\|^2\leq\hat{c}_5 L_1 L_2 C_R^2(\tau^2+h_1^4+h_2^4)^2.\tag{4.19}$$

Next, taking the inner product of (4.8b) with $-\Delta_t\Delta_h e^k$ for the time level $1\leq k\leq N-1$, we have

$$\begin{aligned}&-(\mathcal{A}_h\Delta_t e^k,\Delta_t\Delta_h e^k)\\=&-(1+\mathrm{i}c_1)(\Lambda_h e^{\bar{k}},\Delta_t\Delta_h e^k)-\gamma(\mathcal{A}_h e^{\bar{k}},\Delta_t\Delta_h e^k)\\&+(1+\mathrm{i}c_2)(\mathcal{A}_h(|U^k|^2U^{\bar{k}}-|u^k|^2u^{\bar{k}}),\Delta_t\Delta_h e^k)-(R^k,\Delta_t\Delta_h e^k),\quad 1\leq k\leq N-1.\end{aligned}\tag{4.20}$$

Denote

$$G^{k+1}=(\Lambda_h e^{k+1},\Delta_h e^{k+1})+(\Lambda_h e^k,\Delta_h e^k).$$

Noticing that

$$\begin{aligned}
&-(\mathcal{A}_h\Delta_t e^k,\Delta_t\Delta_h e^k)\geq\frac{1}{3}|\Delta_t e^k|_1^2,\\
&(\Lambda_h e^{\bar{k}},\Delta_t\Delta_h e^k)=\frac{1}{4\tau}(\Lambda_h(e^{k+1}+e^{k-1}),\Delta_h(e^{k+1}-e^{k-1}))\\
&\qquad\qquad\qquad\quad=\frac{1}{4\tau}[(\Lambda_h e^{k+1},\Delta_h e^{k+1})-(\Lambda_h e^{k-1},\Delta_h e^{k-1})]\\
&\qquad\qquad\qquad\quad=\frac{1}{4\tau}(G^{k+1}-G^k)
\end{aligned}$$

and

$$\begin{aligned}
&\left|(\mathcal{A}_h(|U^k|^2U^{\bar{k}}-|u^k|^2u^{\bar{k}}),\Delta_t\Delta_h e^k)\right|\\
\leq&\left|(\mathcal{A}_h(|U^k|^2e^{\bar{k}}),\Delta_t\Delta_h e^k)\right|+\left|(\mathcal{A}_h(U^k\overline{e^k}u^{\bar{k}}),\Delta_t\Delta_h e^k)\right|+\left|(\mathcal{A}_h(e^k\overline{u^k}u^{\bar{k}}),\Delta_t\Delta_h e^k)\right|\\
\leq&\sqrt{2}(3C_u^2+3C_u+2)(\|e^{\bar{k}}\|_\infty+\|e^k\|_\infty+|e^{\bar{k}}|_1+|e^k|_1)|\Delta_t e^k|_1,
\end{aligned}$$

in which Lemma 2.5 is used. Taking the real part of (4.20) and combining Lemmas 2.6 and 2.7, we have

$$\begin{aligned}
&\frac{1}{3}|\Delta_t e^k|_1^2+\frac{1}{4\tau}(G^{k+1}-G^k)\\
\leq&-\gamma\mathrm{Re}(\mathcal{A}_h e^{\bar{k}},\Delta_t\Delta_h e^k)+\sqrt{2}\hat{c}_2\Big[2\hat{c}(\|\Delta_h e^{\bar{k}}\|+\|\Delta_h e^k\|)+\hat{c}(\|e^{\bar{k}}\|+\|e^k\|)\Big]|\Delta_t e^k|_1\\
&\quad-\mathrm{Re}(R^k,\Delta_t\Delta_h e^k)\\
\leq&-\gamma\mathrm{Re}(\mathcal{A}_h e^{\bar{k}},\Delta_t\Delta_h e^k)+2\sqrt{2}\hat{c}\cdot\hat{c}_2\left(\varepsilon_1(\|\Delta_h e^{\bar{k}}\|+\|\Delta_h e^k\|)^2+\frac{1}{4\varepsilon_1}|\Delta_t e^k|_1^2\right)\\
&\quad+\sqrt{2}\hat{c}\cdot\hat{c}_2\left(\varepsilon_2(\|e^{\bar{k}}\|+\|e^k\|)^2+\frac{1}{4\varepsilon_2}|\Delta_t e^k|_1^2\right)-\mathrm{Re}(R^k,\Delta_t\Delta_h e^k),\quad 1\leq k\leq l.
\end{aligned}$$

Taking

$$\frac{2\sqrt{2}\hat{c}\cdot\hat{c}_2}{4\varepsilon_1}=\frac{1}{6},\quad \frac{\sqrt{2}\hat{c}\cdot\hat{c}_2}{4\varepsilon_2}=\frac{1}{6},$$

we have for $1\leq k\leq l$

$$\begin{aligned}
\frac{1}{4\tau}(G^{k+1}-G^k)\leq&-\gamma\mathrm{Re}(\mathcal{A}_h e^{\bar{k}},\Delta_t\Delta_h e^k)+12\hat{c}^2\hat{c}_2^2(\|\Delta_h e^{\bar{k}}\|^2+\|\Delta_h e^k\|^2)\\
&+3\hat{c}^2\hat{c}_2^2(\|e^{\bar{k}}\|^2+\|e^k\|^2)-\mathrm{Re}(R^k,\Delta_t\Delta_h e^k).
\end{aligned}\tag{4.21}$$

Replacing k with n in (4.21) and summing up for n from 1 to k, we have

$$\begin{aligned}\frac{1}{4\tau}(G^{k+1}-G^1)\leq&-\gamma\mathrm{Re}\sum_{n=1}^{k}(\mathcal{A}_h e^{\bar{n}},\Delta_t\Delta_h e^n)+12\hat{c}^2\hat{c}_2^2\sum_{n=1}^{k}(\|\Delta_h e^{\bar{n}}\|^2+\|\Delta_h e^n\|^2)\\&+3\hat{c}^2\hat{c}_2^2\sum_{n=1}^{k}(\|e^{\bar{n}}\|^2+\|e^n\|^2)-\mathrm{Re}\sum_{n=1}^{k}(R^n,\Delta_t\Delta_h e^n),\quad 1\leq k\leq l.\end{aligned}$$

Noticing by Lemma 2.3 that

$$\mathrm{Re}(\mathcal{A}_h e^{n-1},\Delta_h e^{n+1})=\mathrm{Re}(\mathcal{A}_h e^{n+1},\Delta_h e^{n-1}),\quad 1\leq n\leq k,$$

we have

$$\begin{aligned}&\mathrm{Re}\sum_{n=1}^{k}(\mathcal{A}_h e^{\bar{n}},\Delta_t\Delta_h e^n)\\=&\frac{1}{4\tau}\sum_{n=1}^{k}\mathrm{Re}(\mathcal{A}_h(e^{n+1}+e^{n-1}),\Delta_h(e^{n+1}-e^{n-1}))\\=&\frac{1}{4\tau}[\mathrm{Re}(\mathcal{A}_h e^{k+1},\Delta_h e^{k+1})+\mathrm{Re}(\mathcal{A}_h e^{k},\Delta_h e^{k})-\mathrm{Re}(\mathcal{A}_h e^{1},\Delta_h e^{1})]\\\leq&\frac{1}{4\tau}(\|\mathcal{A}_h e^{k+1}\|\cdot\|\Delta_h e^{k+1}\|+\|\mathcal{A}_h e^{k}\|\cdot\|\Delta_h e^{k}\|+\|\mathcal{A}_h e^{1}\|\cdot\|\Delta_h e^{1}\|)\\\leq&\frac{1}{4\tau}(\|e^{k+1}\|\cdot\|\Delta_h e^{k+1}\|+\|e^{k}\|\cdot\|\Delta_h e^{k}\|+\|e^{1}\|\cdot\|\Delta_h e^{1}\|)\\\leq&\frac{1}{4\tau}\left(\frac{1}{4\varepsilon_3}\|e^{k+1}\|^2+\varepsilon_3\|\Delta_h e^{k+1}\|^2+\frac{1}{4\varepsilon_4}\|e^{k}\|^2+\varepsilon_4\|\Delta_h e^{k}\|^2\right)+\frac{1}{2\tau}(\|e^1\|^2+\|\Delta_h e^1\|^2).\end{aligned}$$

In addition, we have

$$\begin{aligned}&\sum_{n=1}^{k}(R^n,\Delta_t\Delta_h e^n)\\=&\frac{1}{2\tau}[(R^k,\Delta_h e^{k+1})+(R^{k-1},\Delta_h e^{k})-(R^2,\Delta_h e^{1})-(R^1,\Delta_h e^{0})]-\sum_{n=2}^{k-1}(\Delta_t R^n,\Delta_h e^n)\\\leq&\frac{1}{2\tau}(\|R^k\|\cdot\|\Delta_h e^{k+1}\|+\|R^{k-1}\|\cdot\|\Delta_h e^{k}\|+\|R^2\|\cdot\|\Delta_h e^{1}\|)+\sum_{n=2}^{k-1}\|\Delta_t R^n\|\cdot\|\Delta_h e^n\|\\\leq&\frac{1}{2\tau}\left(\frac{1}{4\varepsilon_5}\|R^k\|^2+\varepsilon_5\|\Delta_h e^{k+1}\|^2+\frac{1}{4\varepsilon_6}\|R^{k-1}\|^2+\varepsilon_6\|\Delta_h e^{k}\|^2\right)+\frac{1}{\tau}(\|R^2\|^2+\|\Delta_h e^1\|^2)\\&+\frac{1}{2}\sum_{n=2}^{k-1}(\|\Delta_t R^n\|^2+\|\Delta_h e^n\|^2).\end{aligned}$$

Let $\gamma\varepsilon_3+2\varepsilon_5=\frac{1}{3}$, $\gamma\varepsilon_4+2\varepsilon_6=\frac{1}{3}$, we have

$$\begin{aligned}
&\frac{2}{3}(\|\Delta_h e^{k+1}\|^2+\|\Delta_h e^k\|^2)\\
\le& G^1+\frac{|\gamma|}{4\varepsilon_3}\|e^{k+1}\|^2+\frac{|\gamma|}{4\varepsilon_4}\|e^k\|^2+2|\gamma|(\|e^1\|^2+\|\Delta_h e^1\|^2)\\
&+48\hat{c}^2\hat{c}_2^2\tau\sum_{n=1}^{k}(\|\Delta_h e^{\bar{n}}\|^2+\|\Delta_h e^n\|^2)+12\hat{c}^2\hat{c}_2^2\tau\sum_{n=1}^{k}(\|e^{\bar{n}}\|^2+\|e^n\|^2)\\
&+2\tau\sum_{n=2}^{k-1}(\|\Delta_t R^n\|^2+\|\Delta_h e^n\|^2)+\frac{2}{4\varepsilon_5}\|R^k\|^2+\frac{2}{4\varepsilon_6}\|R^{k-1}\|^2+4(\|R^2\|^2+\|\Delta_h e^1\|^2)\\
&+\frac{1}{3}(\|\Delta_h e^{k+1}\|^2+\|\Delta_h e^k\|^2),\quad 1\le k\le l.
\end{aligned}$$

Rearranging the above formula, we have

$$\begin{aligned}
&\frac{1}{3}(\|\Delta_h e^{k+1}\|^2+\|\Delta_h e^k\|^2)\\
\le&\hat{c}_5\tau\sum_{n=1}^{k}(\|\Delta_h e^{\bar{n}}\|^2+\|\Delta_h e^n\|^2)+\hat{c}_6(\tau^2+h_1^4+h_2^4)^2\\
\le&\hat{c}_5\tau\sum_{n=1}^{k}(\|\Delta_h e^{n+1}\|^2+\|\Delta_h e^n\|^2+\|\Delta_h e^{n-1}\|^2)+\hat{c}_6(\tau^2+h_1^4+h_2^4)^2,\quad 1\le k\le l.
\end{aligned}$$

Thus, we have

$$(1-3\hat{c}_5\tau)\|\Delta_h e^{k+1}\|^2\le 3\hat{c}_5\tau\sum_{n=1}^{k}\|\Delta_h e^n\|^2+3\hat{c}_6(\tau^2+h_1^4+h_2^4)^2,\quad 1\le k\le l. \tag{4.22}$$

When $3\hat{c}_5\tau\le\frac{1}{2}$, combining (4.22) with the Gronwall inequality, we have

$$\|\Delta_h e^{k+1}\|^2\le 6\hat{c}_6\exp(6\hat{c}_5 T)(\tau^2+h_1^4+h_2^4)^2\le[\hat{c}_7(\tau^2+h_1^4+h_2^4)]^2,\quad 1\le k\le l.$$

Therefore,

$$\|\Delta_h e^{l+1}\|\le\hat{c}_7(\tau^2+h_1^4+h_2^4),\quad 1\le l\le N-1. \tag{4.23}$$

Step 3: $\|\cdot\|_\infty$ error estimate. Based on (4.16), (4.23) and the interpolation inequality in Lemma 2.7, we have

$$\|e^l\|_\infty\le\hat{c}_8(\tau^2+h_1^4+h_2^4),\quad 1\le l\le N.$$

This completes this proof. □

4.3 Stability

In what follows, we discuss the stability of the linearized compact difference scheme (3.5a)–(3.5c). Let $\{v_{ij}^k \,|\, (i,j)\in\bar{\omega},\ 0\le k\le N\}$ be the solution of the following difference scheme

$$\begin{cases} \mathcal{A}_h\Delta_t v_{ij}^k = (1+\mathrm{i}c_1)\Lambda_h v_{ij}^{\bar{k}} + \gamma\mathcal{A}_h v_{ij}^{\bar{k}} - (1+\mathrm{i}c_2)\mathcal{A}_h(|v_{ij}^k|^2 v_{ij}^{\bar{k}}), & (i,j)\in\bar{\omega}, \quad 1\le k\le N-1, \\ \mathcal{A}_h\delta_t v_{ij}^{\frac{1}{2}} = (1+\mathrm{i}c_1)\Lambda_h v_{ij}^{\frac{1}{2}} + \gamma\mathcal{A}_h v_{ij}^{\frac{1}{2}} - (1+\mathrm{i}c_2)\mathcal{A}_h(|\hat{v}_{ij}^0|^2 v_{ij}^0), & (i,j)\in\bar{\omega}, \\ v_{ij}^0 = u_{ij}^0 + \phi_{ij}^0, & (i,j)\in\bar{\omega}. \end{cases}$$

Denote $\chi_{ij}^k = v_{ij}^k - u_{ij}^k$, and we have the following perturbation system

$$\begin{cases} \mathcal{A}_h\Delta_t\chi_{ij}^k = (1+\mathrm{i}c_1)\Lambda_h\chi_{ij}^{\bar{k}} + \gamma\mathcal{A}_h\chi_{ij}^{\bar{k}} - (1+\mathrm{i}c_2)\mathcal{A}_h(|v_{ij}^k|^2 v_{ij}^{\bar{k}} \\ \qquad\qquad -|u_{ij}^k|^2 u_{ij}^{\bar{k}}), \quad (i,j)\in\bar{\omega}, \quad 1\le k\le N-1, & \text{(4.25a)} \\ \mathcal{A}_h\delta_t\chi_{ij}^{\frac{1}{2}} = (1+\mathrm{i}c_1)\Lambda_h\chi_{ij}^{\frac{1}{2}} + \gamma\mathcal{A}_h\chi_{ij}^{\frac{1}{2}} - (1+\mathrm{i}c_2)\mathcal{A}_h(|\hat{v}_{ij}^0|^2 v_{ij}^0 \\ \qquad\qquad -|\hat{u}_{ij}^0|^2 u_{ij}^0), \quad (i,j)\in\bar{\omega}, & \text{(4.25b)} \\ \chi_{ij}^0 = \phi_{ij}^0, \qquad (i,j)\in\bar{\omega}, & \text{(4.25c)} \end{cases}$$

where

$$|v_{ij}^k|^2 v_{ij}^k - |u_{ij}^k|^2 u_{ij}^k = |v_{ij}^k|^2\chi_{ij}^k + v_{ij}^k u_{ij}^k \overline{\chi}_{ij}^k + |u_{ij}^k|^2\chi_{ij}^k.$$

Theorem 4.3. *Suppose*

$$\|\phi^0\| \le C_1 h_1^{1+\sigma} h_2^{1+\sigma}$$

with C_1, σ being positive constants. Let $\{\chi_{ij}^k \,|\, (i,j)\in\bar{\omega},\ 0\le k\le N\}$ be the solution of the difference scheme (4.25a)–(4.25c). *Denote*

$$C_2 = \sqrt{30}\exp(12\hat{c}_9 T)$$

with

$$\hat{c}_9 = |\gamma| + \sqrt{1+c_2^2}[(C_u+1)^2 + (C_u+1)\cdot(C_u+2) + (C_u+2)^2].$$

When $2C_1C_2h_1^\sigma h_2^\sigma \le 1$, we have

$$\|\chi^k\| \le C_2\|\phi^0\|, \quad 0\le k\le N.$$

Proof. By using (4.25c), we easily know that Theorem 4.3 holds for $k=0$. We will divide the proof into two steps below in order to obtain the stability.

Step 1: Stability in the first level. Taking the inner product of (4.25b) with $\chi^{\frac{1}{2}}$, we have

$$\begin{aligned}(\mathcal{A}_h\delta_t\chi^{\frac{1}{2}},\chi^{\frac{1}{2}})=&(1+\mathrm{i}c_1)(\Lambda_h\chi^{\frac{1}{2}},\chi^{\frac{1}{2}})+\gamma(\mathcal{A}_h\chi^{\frac{1}{2}},\chi^{\frac{1}{2}})\\&-(1+\mathrm{i}c_2)(\mathcal{A}_h(|\hat{v}^0|^2v^0-|\hat{u}^0|^2u^0),\chi^{\frac{1}{2}}).\end{aligned}\tag{4.26}$$

Then, we estimate each term in (4.26).

Noticing

$$\begin{cases}\mathrm{Re}(\mathcal{A}_h\delta_t\chi^{\frac{1}{2}},\chi^{\frac{1}{2}})=\dfrac{1}{2\tau}\left[(\mathcal{A}_h\chi^1,\chi^1)-(\mathcal{A}_h\chi^0,\chi^0)\right], & (4.27\text{a})\\ (\Lambda_h\chi^{\frac{1}{2}},\chi^{\frac{1}{2}})\le-\dfrac{2}{3}|\chi^{\frac{1}{2}}|_1^2, & (4.27\text{b})\\ (\mathcal{A}_h\chi^{\frac{1}{2}},\chi^{\frac{1}{2}})\le\|\chi^{\frac{1}{2}}\|^2, & (4.27\text{c})\\ |(\mathcal{A}_h(|\hat{v}^0|^2v^0-|\hat{u}^0|^2u^0),\chi^{\frac{1}{2}})| & \\ \quad\le\left(\||\hat{u}^0|^2\|_\infty^2+\|\hat{u}^0\hat{v}^0\|_\infty+\||\hat{v}^0|^2\|_\infty^2\right)\|\chi^0\|\cdot\|\chi^{\frac{1}{2}}\|. & (4.27\text{d})\end{cases}$$

Taking the real part of (4.26) and inserting (4.27a)–(4.27d) yield

$$\begin{aligned}&\frac{1}{2\tau}\left[(\mathcal{A}_h\chi^1,\chi^1)-(\mathcal{A}_h\chi^0,\chi^0)\right]\\ \le&|\gamma|\|\chi^{\frac{1}{2}}\|^2+\sqrt{1+c_2^2}\left(\||\hat{u}^0|^2\|_\infty^2+\|\hat{u}^0\hat{v}^0\|_\infty+\||\hat{v}^0|^2\|_\infty^2\right)\|\chi^0\|\cdot\|\chi^{\frac{1}{2}}\|\\ \le&\hat{c}_8\|\chi^0\|^2+\hat{c}_8\|\chi^1\|^2\\ \le&3\hat{c}_8(\mathcal{A}_h\chi^1,\chi^1)+3\hat{c}_8(\mathcal{A}_h\chi^0,\chi^0).\end{aligned}$$

Thus,

$$(1-6\hat{c}_8\tau)(\mathcal{A}_h\chi^1,\chi^1)\le(1+6\hat{c}_8\tau)(\mathcal{A}_h\chi^0,\chi^0).$$

When $6\hat{c}_8\tau\le\frac{1}{2}$, we have

$$\frac{1}{3}\|\chi^1\|^2\le(\mathcal{A}_h\chi^1,\chi^1)\le(1+24\hat{c}_8\tau)(\mathcal{A}_h\chi^0,\chi^0)\le3\|\chi^0\|^2.$$

Therefore,

$$\|\chi^1\|^2\le9\|\chi^0\|^2.\tag{4.28}$$

Step 2: Stability for $1\le k\le l\le N$. Now, we assume that the conclusion holds for $1\le k\le l\le N-1$. Therefore, we have

$$\|\chi^k\|_\infty^2\le4h_1^{-1}h_2^{-1}\|\chi^k\|^2\le4C_2^2C_1^2h_1^\sigma h_2^\sigma\le1,\quad 1\le k\le l.\tag{4.29}$$

According to induction hypothesis, we have

$$\|u^k\|^2 \le (C_u+1)^2, \quad 1\le k\le l.$$

Therefore, we have

$$\|v^k\|_\infty = \|(v^k-u^k)+u^k\|_\infty \le \|\chi^k\|_\infty + \|u^k\|_\infty \le C_u+2, \quad 1\le k\le l.$$

Next, taking the inner product of (4.25a) with $\chi^{\bar{k}}$, we have

$$\begin{aligned}(\mathcal{A}_h\Delta_t\chi^k,\chi^{\bar{k}}) =& (1+\mathrm{i}c_1)(\Lambda_h\chi^{\bar{k}},\chi^{\bar{k}})+\gamma(\mathcal{A}_h\chi^{\bar{k}},\chi^{\bar{k}})\\ &-(1+\mathrm{i}c_2)(\mathcal{A}_h(|v^k|^2v^{\bar{k}}-|u^k|^2u^{\bar{k}}),\chi^{\bar{k}}).\end{aligned} \tag{4.30}$$

Noticing

$$\begin{cases}\mathrm{Re}(\mathcal{A}_h\Delta_t\chi^k,\chi^{\bar{k}}) = \dfrac{1}{4\tau}\big[(\mathcal{A}_h\chi^{k+1},\chi^{k+1})-(\mathcal{A}_h\chi^{k-1},\chi^{k-1})\big], & (4.31\mathrm{a})\\ (\Lambda_h\chi^{\bar{k}},\chi^{\bar{k}}) \le -\dfrac{2}{3}|\chi^{\bar{k}}|_1^2, & (4.31\mathrm{b})\\ (\mathcal{A}_h\chi^{\bar{k}},\chi^{\bar{k}}) \le \|\chi^{\bar{k}}\|^2, & (4.31\mathrm{c})\end{cases}$$

and

$$\begin{aligned}&\Big(\mathcal{A}_h(|v^k|^2v^{\bar{k}}-|u^k|^2u^{\bar{k}}),\chi^{\bar{k}}\Big)\\ \le& \big|(\mathcal{A}_h|u^k|^2\chi^{\bar{k}},\chi^{\bar{k}})\big| + \big|(\mathcal{A}_hu^kv^k\overline{\chi^{\bar{k}}},\chi^{\bar{k}})\big| + \big|(\mathcal{A}_h|v^k|^2\chi^{\bar{k}},\chi^{\bar{k}})\big|\\ \le& \sqrt{2}\big[(C_u+1)^2+(C_u+1)(C_u+2)+(C_u+2)^2\big]\|\chi^{\bar{k}}\|^2.\end{aligned} \tag{4.32}$$

Thus, taking the real part of (4.30) and substituting (4.31a)–(4.32) into (4.30), we have

$$\frac{1}{4\tau}\big[(\mathcal{A}_h\chi^{k+1},\chi^{k+1})-(\mathcal{A}_h\chi^{k-1},\chi^{k-1})\big] \le \hat{c}_9\|\chi^{\bar{k}}\|^2. \tag{4.33}$$

Let

$$H^{k+1} = (\mathcal{A}_h\chi^{k+1},\chi^{k+1})+(\mathcal{A}_h\chi^k,\chi^k),$$

(4.33) becomes

$$H^{k+1}-H^k \le 4\hat{c}_9\tau\|\chi^{\bar{k}}\|^2 \le 6\hat{c}_9\tau(H^{k+1}+H^k).$$

In other words,

$$(1-6\hat{c}_9\tau)H^{k+1} \le (1+6\hat{c}_9\tau)H^k.$$

When $6\hat{c}_9\tau\le\frac{1}{2}$, we have

$$H^{k+1}\le(1+24\hat{c}_9\tau)H^k.$$

In combination of the Gronwall inequality with (4.28), we have

$$\begin{aligned}H^{k+1}&\le\exp(24\hat{c}_9k\tau)H^1\\&\le\exp(24\hat{c}_9T)\left[(\mathcal{A}_h\chi^1,\chi^1)+(\mathcal{A}_h\chi^0,\chi^0)\right]\\&\le\exp(24\hat{c}_9T)\left[\|\chi^1\|^2+\|\chi^0\|^2\right]\\&\le10\exp(24\hat{c}_9T)\|\phi^0\|^2.\end{aligned}$$

According to Lemma 2.2, we have

$$\|\chi^{k+1}\|^2\le3H^{k+1}\le30\exp(24\hat{c}_9T)\|\phi^0\|^2.$$

This completes this proof. □

5 Numerical experiment

To verify the accuracy and stability of the difference scheme (3.5a)–(3.5c), three numerical examples are test.

Example 5.1. Consider the following KT equation

$$u_t=(1+\mathrm{i})\Delta u+u-(1+\mathrm{i})|u|^2u+f(x,y,t),\quad (x,y)\in(0,L)^2,\quad 0<t\le1.$$

$f(x,y,t)$, initial and boundary conditions are determined by the exact solution

$$u(x,y,t)=\cos(\pi x)\cos(\pi y)\exp(-\mathrm{i}t).$$

• **Convergence test in space**. Firstly, we test the spatial convergence orders of the compact difference scheme (3.5a)–(3.5c) with $L=1$, namely, fixing the temporal step and reducing spatial step half by half ($h=1/4$, $1/8$, $1/16$, $1/32$, $1/64$), here we take $h=h_1=h_2$.

In Table 1, we list the L^2-norm errors and L^∞-norm errors and their corresponding spatial convergence orders. As we observe from Table 1, the scheme (3.5a)–(3.5c) is of order four approximately.

• **Convergence test in time**. Then we test the temporal convergence order with $L=1$. We set $h=h_1=h_2=1/128$ and reduce the temporal step τ half by half, respectively. Numerical results are shown in Table 2, which is consistent with the theoretical results.

Table 1: Example 5.1: L^2-norm and L^∞-norm errors behavior versus space-grid size reduction with $\tau=1/10000$.

h	$E_2(h,\tau)$	Ord_2^h	$E_\infty(h,\tau)$	Ord_∞^h
1/4	1.1728e−3	–	1.5485e−3	–
1/8	6.0096e−5	4.2866	9.5175e−5	4.0242
1/16	3.3685e−6	4.1571	5.9217e−6	4.0065
1/32	1.9877e−7	4.0830	3.6976e−7	4.0014
1/64	1.2098e−8	4.0383	2.3179e−8	3.9957

Table 2: Example 5.1: L^2-norm and L^∞-norm errors behavior versus time-grid size reduction with $h=1/128$.

τ	$E_2(h,\tau)$	Ord_2^τ	$E_\infty(h,\tau)$	Ord_∞^τ
1/20	2.3552e−5	–	6.5360e−5	–
1/40	3.7761e−6	2.6409	6.7088e−6	3.2843
1/80	9.4326e−7	2.0012	1.8427e−6	1.8641
1/160	2.3619e−7	1.9977	4.6141e−7	1.9978
1/320	5.9431e−8	1.9906	1.1610e−7	1.9907

• **Stability test.** Next, we test the stability of the compact difference scheme (3.5a)–(3.5c). The numerical results are demonstrated in Fig. 1. Each error curve in Fig. 1 is obtained by the reduced spatial step $h=1/2$, $1/4$, $1/8$, $1/16$, $1/32$ and the fixed temporal step size. We observe that the numerical errors of each curve tend

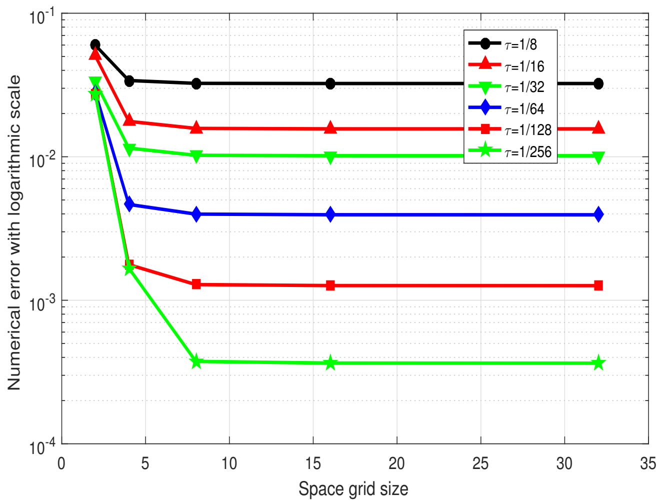
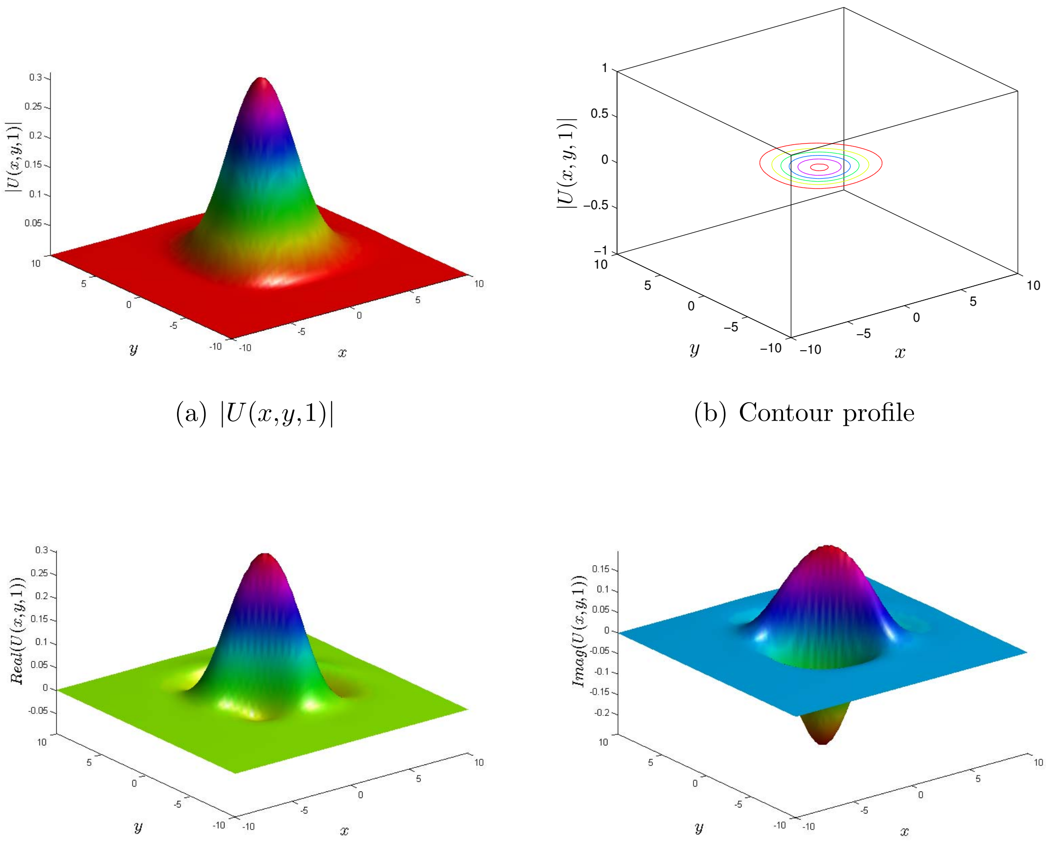


Figure 1: Example 5.1: The chart of the unconditional stability test.



(a) $|U(x,y,1)|$ (b) Contour profile

(c) Real(U) (d) Imag(U)

Figure 2: Example 5.3: (a) the modulus of numerical solution U; (b) the contour profile of the numerical solution U; (c) the real part of U; (d) the imaginary part of U. In the calculation, we take $L=10$, $t=1$, $M_1=M_2=60$ and $N=100$.

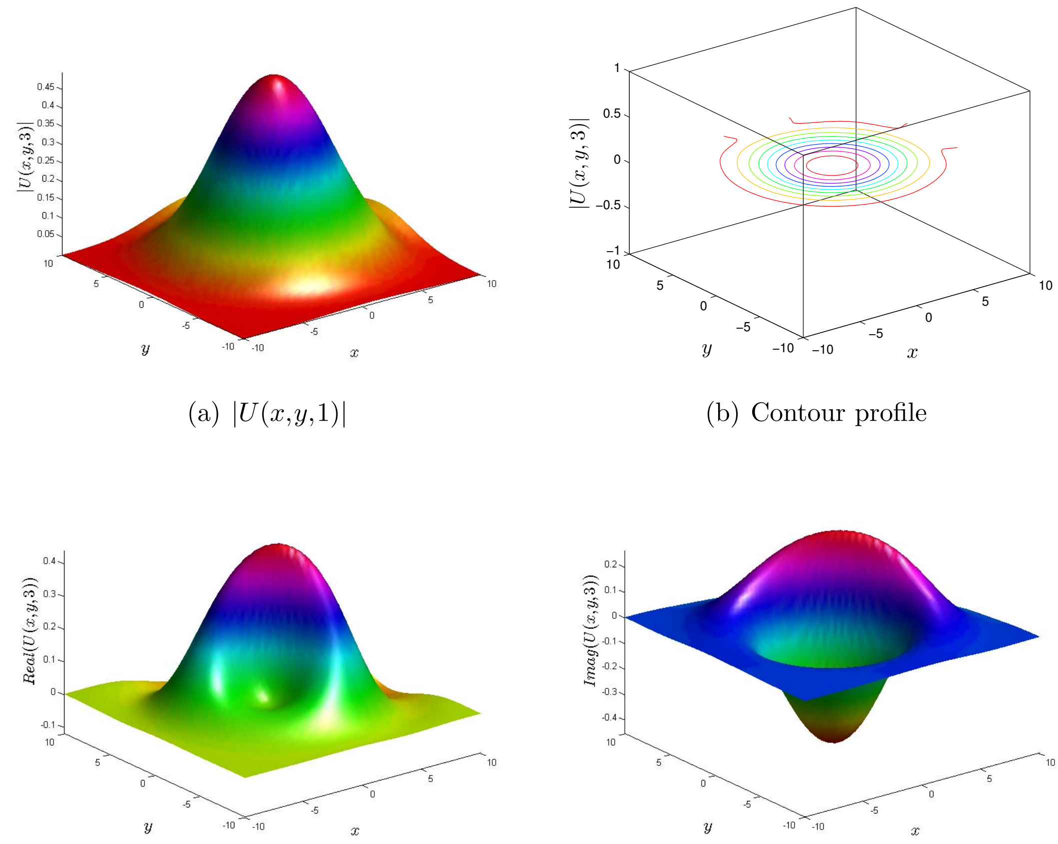
to a fixed value as the spatial grids refine gradually. The main reason is that the control error under this situation originates mainly from the temporal discretization. This demonstrates that the compact difference scheme (3.5a)–(3.5c) is stable and independent of the step ratio.

Example 5.2. Then we consider a problem below, in which the exact solution is unknown.

$$u_t=(1+0.5\mathrm{i})\Delta u+\gamma u-(1+\mathrm{i})|u|^2u, \quad (x,y)\in(-L,L)^2, \quad 0<t\leq 1.$$

The corresponding initial condition is Gaussian pulse

$$\varphi(x,y)=\exp(-2(x^2+y^2))\exp(-\mathrm{i}S_0(x,y))$$



(a) $|U(x,y,1)|$ (b) Contour profile

(c) Real(U) (d) Imag(U)

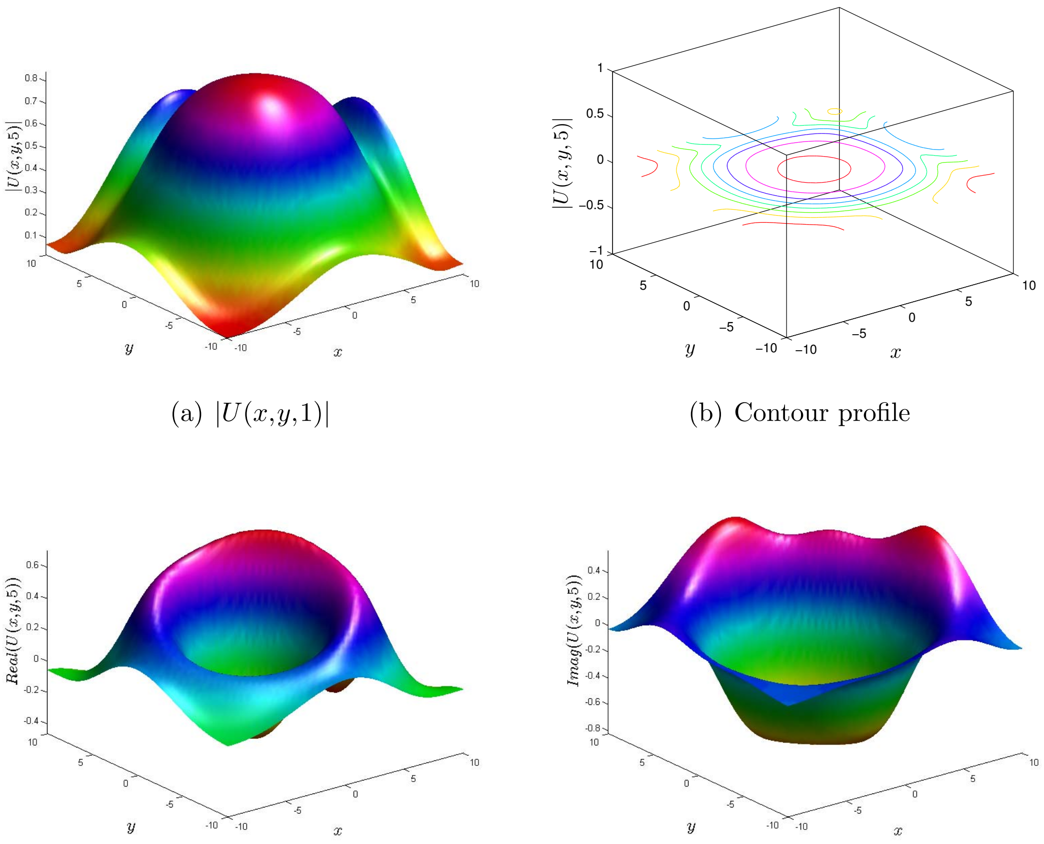
Figure 3: Example 5.3: (a) the modulus of numerical solution U; (b) the contour profile of the numerical solution U; (c) the real part of U; (d) the imaginary part of U. In the calculation, we take $L=10$, $t=3$, $M_1=M_2=60$ and $N=100$.

with

$$S_0(x,y)=1/(\exp(x+y)+\exp(-(x+y))).$$

Here we take $L=4$ and $\gamma=-5,0,5$, respectively.

Convergence test in space and time. In Table 3, we see that all the convergence orders in space are fourth-order accuracy whatever the parameter γ is negative, zero, or positive. In Table 4, convergence in time is tested and the convergence order is approximately second-order accurate. The parameter γ determines the degree of the aggregation and divergence for the solutions.



(a) $|U(x,y,1)|$ (b) Contour profile

(c) Real(U) (d) Imag(U)

Figure 4: Example 5.3: (a) The modulus of numerical solution U; (b) The contour profile of the numerical solution U; (c) The real part of U; (d) The imaginary part of U. In the calculation, we take $L=10$, $t=5$, $M_1=M_2=60$ and $N=100$.

Example 5.3. We finally simulate a KT problem (1.1) with an initial value

$$\varphi(x,y)=\operatorname{sech}(x)\operatorname{sech}(y)\exp(\mathrm{i}(x+y))$$

on a large domain $\Omega=[-10,10]^2$. The coefficients in the problem are chosen as $c_1=c_2=\gamma=1$. The exact solution is unknown.

Convergence test in space and time. The numerical results with $t=1$ are listed in Table 5 and Table 6. Similar convergence orders are demonstrated even if the computation is on a larger domain with a rough grid.

Simulation at different time. We simulate the evolution of the solution at the time $t=1,3,5$, and clear evolution surfaces are shown, respectively, in Figs. 2–4.

Table 3: Example 5.2: L^2-norm and L^∞-norm errors versus space-grid size reduction with $\tau=1/16$.

	h	$E_2(h,\tau)$	Ord_2^h	$E_\infty(h,\tau)$	Ord_∞^h
$\gamma=-5$	2/5	–	–	–	–
	1/5	1.0567e−4	–	9.1639e−5	–
	1/10	8.0917e−6	3.7069	7.4098e−6	3.6284
	1/20	4.8783e−7	4.0520	4.7790e−7	3.9547
	1/40	3.0072e−8	4.0199	3.0182e−8	3.9850
	1/80	1.8725e−9	4.0054	1.8811e−9	4.0040
$\gamma=0$	2/5	–	–	–	–
	1/5	1.1663e−4	–	8.7459e−5	–
	1/10	8.6631e−6	3.7510	7.8130e−6	3.4847
	1/20	5.2384e−7	4.0477	4.6997e−7	4.0553
	1/40	3.2317e−8	4.0188	2.9956e−8	3.9717
	1/80	2.0122e−9	4.0055	1.8702e−9	4.0016
$\gamma=5$	2/5	–	–	–	–
	1/5	1.2768e−2	–	5.3820e−3	–
	1/10	7.2237e−4	4.1436	3.0621e−4	4.1356
	1/20	4.4182e−5	4.0312	1.8800e−5	4.0257
	1/40	2.7430e−6	4.0096	1.1696e−6	4.0067
	1/80	1.7105e−7	4.0033	7.3013e−8	4.0017

These results indicate the present compact difference scheme can perform well even in the long time simulation.

6 Concluding remarks

In summary we numerically study a linearized difference scheme for solving the KT equation in two dimensions under the Neumann boundary condition. The pointwise error estimate and stability for the present scheme are proved at length. Several numerical examples with nice portraits demonstrate good performance of the high-order compact scheme.

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We would like to thank the anonymous and excellent referees for the constructive comments and valuable suggestions, which greatly improved the original manuscript.

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Table 4: Example 5.2: L^2-norm and L^∞-norm errors versus space-grid size reduction with $h=1/16$.

	τ	$E_2(h,\tau)$	Ord_2^τ	$E_\infty(h,\tau)$	Ord_∞^τ
$\gamma=-5$	1/20	–	–	–	–
	1/40	2.0249e−4	–	1.1163e−4	–
	1/80	5.1202e−5	1.9835	2.2375e−5	2.3187
	1/160	1.2851e−5	1.9943	5.6613e−6	1.9827
	1/320	3.2133e−6	1.9998	1.4185e−6	1.9968
	1/640	8.0304e−7	2.0005	3.5471e−7	1.9996
$\gamma=0$	1/20	–	–	–	–
	1/40	7.1771e−4	–	1.5421e−4	–
	1/80	1.9132e−4	1.9074	4.1365e−5	1.8984
	1/160	4.9477e−5	1.9511	1.0715e−5	1.9487
	1/320	1.2556e−5	1.9783	2.7196e−6	1.9782
	1/640	3.1599e−6	1.9905	6.8435e−7	1.9906
$\gamma=5$	1/20	–	–	–	–
	1/40	6.4187e−1	–	1.3329e−1	–
	1/80	1.7541e−1	1.8716	4.3395e−2	1.6190
	1/160	4.7526e−2	1.8839	1.3616e−2	1.6722
	1/320	1.2422e−2	1.9358	3.7499e−3	1.8604
	1/640	3.1772e−3	1.9671	9.8047e−4	1.9353

Table 5: Example 5.3: L^2-norm and L^∞-norm errors versus space-grid size reduction with $\tau=1/4$.

h	$E_2(h,\tau)$	Ord_2^h	$E_\infty(h,\tau)$	Ord_∞^h
4/3	–	–	–	–
2/3	1.6973e−1	–	7.5635e−2	–
1/3	1.1359e−2	3.9013	7.7754e−3	3.2821
1/6	6.3896e−4	4.1520	4.1838e−4	4.2160
1/12	3.8885e−5	4.0384	2.6130e−5	4.0010
1/24	2.4299e−6	4.0003	1.6174e−6	4.0139

Table 6: Example 5.3: L^2-norm and L^∞-norm errors versus time-grid size reduction with $h=5/32$.

τ	$E_2(h,\tau)$	Ord_2^τ	$E_\infty(h,\tau)$	Ord_∞^τ
1/10	–	–	–	–
1/20	2.8548e−2	–	7.0178e−3	–
1/40	6.3852e−3	2.1606	1.3167e−3	2.4141
1/80	1.5849e−3	2.0103	3.4139e−4	1.9474
1/160	3.9649e−4	1.9990	8.7107e−5	1.9705
1/320	9.9255e−5	1.9981	2.2025e−5	1.9837

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