Ann. Appl. Math.
doi: 10.4208/aam.OA-2023-0006

Vol. **39**, No. 1, pp. 79-98
February 2023

U-Eigenvalues' Inclusion Sets of Complex Tensors

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Received 7 February 2023; Accepted (in revised version) 28 February 2023

Abstract. In this paper, we study some inclusion sets of US-eigenvalues and U-eigenvalues based on quantum information. We give three inclusion sets theorems of US-eigenvalues and two inclusion sets theorems of U-eigenvalues. And we obtain the relationships among these inclusion sets. Some numerical examples are shown to illustrate the conclusions.

AMS subject classifications: 15A18, 15A69

Key words: Complex tensor, US-eigenvalue, U-eigenvalue, inclusion set.

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1 Introduction

Let n be a positive integer and $[n]=\{1,2,\cdots,n\}$. Call

$$\mathcal{A}=(a_{i_1i_2\cdots i_d}) \quad \text{for all} \quad a_{i_1i_2\cdots i_d}\in\mathbb{C}, \quad i_k\in[n_k], \quad k\in[d],$$

a d-order $(n_1\times n_2\times\cdots\times n_d)$-dimensional complex tensor. When $n_1=n_2=\cdots=n_d=n$, $\mathcal{A}$ is a d-order n-dimensional complex tensor. In particular, when $d=1$ and $d=2$, they are vector and matrix, respectively. Let $\mathbb{C}^{n_1\times n_2\times\cdots\times n_d}$ be the set of d-order $(n_1\times n_2\times\cdots\times n_d)$-dimensional tensors over $\mathbb{C}$.

In 2014, Ni et al. [1] proposed definitions of U-eigenvalues and US-eigenvalues based on quantum information, i.e., converting the geometric measure of the entanglement [2–4] problem to an algebraic equation system problem. Using an iterative

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algorithm, Che et al. [14] computed the U- and US-eigenpairs of complex tensors in 2017. In 2018, Che et al. [15] studied the geometric measures of entanglement in multipartite pure states via complex-valued neural networks. Due to the complexity of tensor operations, it is troublesome to computing the U- and US-eigenvalues of complex tensors. Sometimes, we only need to know the range of them. Therefore, the inclusion sets of U- and US- eigenvalues are given in this paper.

For $\mathcal{A},\mathcal{B}\in\mathbb{C}^{n_1\times n_2\times\cdots\times n_d}$, the inner product and norm are

$$\begin{aligned}&\langle\mathcal{A},\mathcal{B}\rangle=\sum_{i_1,i_2,\cdots,i_d=1}^{n_1,n_2,\cdots,n_d}(\mathcal{A}^*)_{i_1i_2\cdots i_d}(\mathcal{B})_{i_1i_2\cdots i_d},\\&\|\mathcal{A}\|=\sqrt{\langle\mathcal{A},\mathcal{A}\rangle},\end{aligned}$$

where $(\mathcal{A}^*)_{i_1i_2\cdots i_d}$ denotes the complex conjugate of $(\mathcal{A})_{i_1i_2\cdots i_d}$. A rank-one tensor is defined as $\otimes_{i=1}^d\boldsymbol{x}^{(i)}\in\mathbb{C}^{n_1\times n_2\times\cdots\times n_d}$, where $\boldsymbol{x}^{(i)}\in\mathbb{C}^{n_i},i\in[d]$. By tensor product,

$$\begin{aligned}&\mathcal{A}^*\left(\boldsymbol{x}^{(1)},\cdots,\boldsymbol{x}^{(k-1)},I_{n_k},\boldsymbol{x}^{(k+1)},\cdots,\boldsymbol{x}^{(d)}\right),\\&\mathcal{A}\left(\boldsymbol{x}^{(1)*},\cdots,\boldsymbol{x}^{(k-1)*},I_{n_k},\boldsymbol{x}^{(k+1)*},\cdots,\boldsymbol{x}^{(d)*}\right),\end{aligned}$$

for vectors $\boldsymbol{x}^{(i)}\in\mathbb{C}^{n_i}\,(i\in[d])$ denote vectors in $\mathbb{C}^{n_k}$, whose pth components are

$$\begin{aligned}&\left(\mathcal{A}^*\left(\boldsymbol{x}^{(1)},\cdots,\boldsymbol{x}^{(k-1)},I_{n_k},\boldsymbol{x}^{(k+1)},\cdots,\boldsymbol{x}^{(d)}\right)\right)_p\\&=\sum_{i_1,\cdots,i_{k-1},i_{k+1},\cdots,i_d=1}^{n_1,\cdots,n_{k-1},n_{k+1},\cdots,n_d}(\mathcal{A}^*)_{i_1\cdots i_{k-1}pi_{k+1}\cdots i_d}x_{i_1}^{(1)}\cdots x_{i_{k-1}}^{(k-1)}x_{i_{k+1}}^{(k+1)}\cdots x_{i_d}^{(d)}, &&(1.1a)\\&\left(\mathcal{A}\left(\mathbf{x}^{(1)*},\cdots,\boldsymbol{x}^{(k-1)*},I_{n_k},\boldsymbol{x}^{(k+1)*},\cdots,\boldsymbol{x}^{(d)*}\right)\right)_p\\&=\sum_{i_1,\cdots,i_{k-1},i_{k+1},\cdots,i_d=1}^{n_1,\cdots,n_{k-1},n_{k+1},\cdots,n_d}(\mathcal{A})_{i_1\cdots i_{k-1}pi_{k+1}\cdots i_d}x_{i_1}^{(1)*}\cdots x_{i_{k-1}}^{(k-1)*}x_{i_{k+1}}^{(k+1)*}\cdots x_{i_d}^{(d)*}, &&(1.1b)\end{aligned}$$

where I_{n_k} is a $n_k\times n_k$ identity matrix, $p\in[n_k]$, $k\in[d]$.

A tensor $\mathcal{S}=(s_{i_1i_2\cdots i_d})\in\mathbb{C}^{n\times n\times\cdots\times n}$ is called complex symmetric if its entries $s_{i_1i_2\cdots i_d}$ are invariant under any permutation of their indices. Let $\boldsymbol{x}\in\mathbb{C}^n$, similarly,

$$\begin{aligned}&\mathcal{S}^*(I_n,\boldsymbol{x},\cdots,\boldsymbol{x})\in\mathbb{C}^n,\\&\mathcal{S}(I_n,\boldsymbol{x}^*,\cdots,\boldsymbol{x}^*)\in\mathbb{C}^n,\end{aligned}$$

whose pth components are

$$(\mathcal{S}^*(I_n,\boldsymbol{x},\cdots,\boldsymbol{x}))_p=\sum_{i_2,\cdots,i_d=1}^{n}\mathcal{S}^*_{pi_2\cdots i_d}x_{i_2}\cdots x_{i_d}, \tag{1.2a}$$

$$(\mathcal{S}(I_n,\boldsymbol{x}^*,\cdots,\boldsymbol{x}^*))_p=\sum_{i_2,\cdots,i_d=1}^{n}\mathcal{S}_{pi_2\cdots i_d}x^*_{i_2}\cdots x^*_{i_d}, \tag{1.2b}$$

where I_n is a $n\times n$ identity matrix, $p\in[n]$.

Definition 1.1 ([1])**.** *Let $\mathcal{A}\in\mathbb{C}^{n_1\times n_2\times\cdots\times n_d}$. We call a number $\lambda\in\mathbb{C}$ an U-eigenvalue of $\mathcal{A}$ and a rank-one tensor*

$$\otimes_{i=1}^{d}\boldsymbol{x}^{(i)}\in\mathbb{C}^{n_1\times n_2\times\cdots\times n_d}\left(\boldsymbol{x}^{(i)}\in\mathbb{C}^{n_i},i\in[d]\right)$$

an U-eigenvector pairs if λ and the rank-one tensor $\otimes_{i=1}^{d}\boldsymbol{x}^{(i)}$ are solutions of the following equations:

$$\begin{cases}\mathcal{A}^*\left(\boldsymbol{x}^{(1)},\cdots,\boldsymbol{x}^{(k-1)},I_k,\boldsymbol{x}^{(k+1)},\cdots,\boldsymbol{x}^{(d)}\right)=\lambda\boldsymbol{x}^{(k)*}, & (1.3a)\\ \mathcal{A}\left(\boldsymbol{x}^{(1)*},\cdots,\boldsymbol{x}^{(k-1)*},I_k,\boldsymbol{x}^{(k+1)*},\cdots,\boldsymbol{x}^{(d)*}\right)=\lambda\boldsymbol{x}^{(k)}, & (1.3b)\\ \left\|\boldsymbol{x}^{(k)}\right\|=1, \end{cases}$$

where $k\in[d]$.

Definition 1.2 ([1])**.** *Let $\mathcal{S}\in\mathbb{C}^{n\times n\times\cdots\times n}$. We call a number $\lambda\in\mathbb{C}$ an US-eigenvalue of $\mathcal{S}$ and a nonzero vector $\boldsymbol{x}\in\mathbb{C}^n$ an US-eigenvector if λ and the nonzero vector $\boldsymbol{x}$ are solutions of the following equations:*

$$\begin{cases}\mathcal{S}^*(I_n,\boldsymbol{x},\cdots,\boldsymbol{x})=\lambda\boldsymbol{x}^*, & (1.4a)\\ \mathcal{S}(I_n,\boldsymbol{x}^*,\cdots,\boldsymbol{x}^*)=\lambda\boldsymbol{x}, & (1.4b)\\ \|\boldsymbol{x}\|=1. \end{cases}$$

The well known Geršgorin-type, Brauer-type and Brualdi-type eigenvalue inclusion sets of matrices were introduced in [5, 6] and [7], respectively. In 2005, L. Q. Qi [8] showed the Geršgorin-type inclusion set of real symmetric tensors, which also holds for general tensors [9]. In [10], C. Q. Li gave the Brauer-type eigenvalue inclusion set of tensors. Using graph theory, C. J. Bu obtained the Brualdi-type eigenvalue inclusion set of square tensors in [11]. There are also many generalizations of these results, see [10, 12].

2 Preparation of manuscript

Let Γ be a digraph with vertex set V and arc set E. A circuit γ of Γ is a sequence $v_{i_1},\cdots,v_{i_p},v_{i_{p+1}}=v_{i_1}$, where $p\geq 2$, $v_{i_1},\cdots,v_{i_p}\in V$ are distinct, and $(v_{i_1},v_{i_2}),(v_{i_2},v_{i_3}),\cdots,\left(v_{i_p},v_{i_1}\right)\in E$. Γ is called weakly connected if for each vertex $v_i\in V$, there exists a circuit such that v_i belongs to the circuit. For $v_i\in V$, let $\Gamma^+(v_i)=\{v_j\in V:(v_i,v_j)\in E\}$. A pre-order defined on V satisfies (1) $v_i\leq v_i$; (2) $v_i\leq v_j$ and $v_j\leq v_k$ implies $v_i\leq v_k$; (3) $v_i\leq v_j$ and $v_j\leq v_i$ cannot conclude $v_i=v_j$, where $v_i,v_j,v_k\in V$ [7].

For a tensor $\mathcal{S}=(s_{i_1i_2\cdots i_d})\in\mathbb{C}^{n\times n\times\cdots\times n}$, we associate with $\mathcal{S}$ a digraph $\Gamma_{\mathcal{S}}$ as follows. The vertex set of $\Gamma_{\mathcal{S}}$ is $V(\mathcal{S})=\{v_1,v_2,\cdots,v_n\}$, the arc set of $\Gamma_{\mathcal{S}}$ is

$$E(\mathcal{S})=\{(v_i,v_j):s_{ii_2\cdots i_d}\neq 0,\ j\in\{i_2,\cdots,i_d\}\neq\{i,\cdots,i\}\}.$$

Lemma 2.1 ([7]). *Let Γ be a digraph. A pre-order is defined on its vertex set. If $\Gamma^+(v)$ is nonempty for each vertex v, then there exists a circuit $v_{i_1},\cdots,v_{i_k},v_{i_{k+1}}=v_{i_1}$ such that $v_{i_{j+1}}$ is a maximal element of $\Gamma^+\left(v_{i_j}\right)$ for $j\in[k]$.*

Lemma 2.2. *Let $a_1,a_2,\cdots,a_n$ be non-negative real numbers and $R_1,R_2,\cdots,R_n$ be positive real numbers and r_k be the maximum root of equation*

$$f_k(x)=(x-a_1)\cdots(x-a_k)-R_1\cdots R_k=0,$$

where $k\in[n]$. Then the following hold:

(1) If $a_1+R_1\geq a_2+R_2\geq\cdots\geq a_n+R_n$, then

$$r_1=a_1+R_1,\ a_2+R_2\leq r_2\leq r_1,\ \cdots,\ a_n+R_n\leq r_n\leq r_{n-1}.$$

$r_n=\cdots=r_1$ if and only if $a_1+R_1=\cdots=a_n+R_n$.

(2) If $a_1+R_1\leq a_2+R_2\leq\cdots\leq a_n+R_n$, then

$$r_1=a_1+R_1,\ r_1\leq r_2\leq a_2+R_2,\ \cdots,\ r_{n-1}\leq r_n\leq a_n+R_n.$$

$r_n=\cdots=r_1$ if and only if $a_1+R_1=\cdots=a_n+R_n$.

Proof. (1) When $n=1$, it clearly holds. When $n=2$, we have $f_2'(x)=2x-a_1-a_2$. Since

$$\begin{aligned}
&f_2(a_2+R_2)=(a_2+R_2-a_1)R_2-R_1R_2=(a_2+R_2-a_1-R_1)R_2\leq 0,\\
&f_2(r_1)=R_1(r_1-a_2)-R_1R_2=R_1(a_1+R_1-a_2-R_2)\geq 0,\\
&f_2'(r_1)=2r_1-a_1-a_2=a_1+2R_1-a_2=a_1+R_1-a_2-R_2+R_1+R_2>0,
\end{aligned}$$

we obtain that $a_2+R_2\le r_2\le r_1$. When $n=3$, assume $r_3<a_3+R_3$, then, we have $f_3(a_3+R_3)>0$. But

$$f_3(a_3+R_3)=(a_3+R_3-a_1)(a_3+R_3-a_2)R_3-R_1R_2R_3\le R_1R_2R_3-R_1R_2R_3=0,$$

which is a contradiction. So we have $r_3\ge a_3+R_3$. Since

$$\begin{aligned}f_3(r_2)=&(r_2-a_1)(r_2-a_2)(r_2-a_3)-R_1R_2R_3\\=&R_1R_2(r_2-a_3-R_3)\\\ge&R_1R_2(a_2+R_2-a_3-R_3)\ge 0,\end{aligned}$$

and

$$f_2(r_3)=(r_3-a_1)(r_3-a_2)-R_1R_2=\frac{R_1R_2R_3}{r_3-a_3}-R_1R_2=R_1R_2\frac{a_3+R_3-r_3}{r_3-a_3}\le 0,$$

we obtain that $r_3\le r_2$. Suppose that $a_k+R_k\le r_k\le r_{k-1}$ holds when $n=k$. Consider the case of $n=k+1$. Assume that $r_{k+1}<a_{k+1}+R_{k+1}$. Then we have $f_{k+1}(a_{k+1}+R_{k+1})>0$. But

$$\begin{aligned}f_{k+1}(a_{k+1}+R_{k+1})=&(a_{k+1}+R_{k+1}-a_1)\cdots(a_{k+1}+R_{k+1}-a_k)R_{k+1}-R_1\cdots R_kR_{k+1}\\\le&R_1\cdots R_kR_{k+1}-R_1\cdots R_kR_{k+1}=0,\end{aligned}$$

which is a contradiction. So we have $r_{k+1}\ge a_{k+1}+R_{k+1}$. Since

$$\begin{aligned}f_{k+1}(r_k)=&(r_k-a_1)\cdots(r_k-a_k)(r_k-a_{k+1})-R_1\cdots R_kR_{k+1}\\=&R_1\cdots R_k(r_k-a_{k+1}-R_{k+1})\\\ge&R_1\cdots R_k(a_k+R_k-a_{k+1}-R_{k+1})\ge 0,\\f_k(r_{k+1})=&(r_{k+1}-a_1)\cdots(r_{k+1}-a_k)-R_1\cdots R_k\\=&\frac{R_1\cdots R_kR_{k+1}}{r_{k+1}-a_{k+1}}-R_1\cdots R_k\\=&R_1\cdots R_k\frac{a_{k+1}+R_{k+1}-r_{k+1}}{r_{k+1}-a_{k+1}}\le 0,\end{aligned}$$

we obtain that $r_{k+1}\le r_k$. By mathematical induction, the result holds.

If $r_n=\cdots=r_1=r$, then

$$\begin{aligned}&f_n(r_n)=f_n(r)=(r-a_1)\cdots(r-a_{n-1})(r-a_n)-R_1\cdots R_{n-1}R_n=0,\\&f_{n-1}(r_{n-1})=f_{n-1}(r)=(r-a_1)\cdots(r-a_{n-1})-R_1\cdots R_{n-1}=0.\end{aligned}$$

It yields $r=a_n+R_n$. Since

$$a_{n-1}+R_{n-1}\le r_{n-1}=r_n=a_n+R_n \quad \text{and} \quad a_{n-1}+R_{n-1}\ge a_n+R_n,$$

we get that

$$a_{n-1}+R_{n-1}=a_n+R_n=r_{n-1}=r_n.$$

Also,

$$a_{n-2}+R_{n-2}\le r_{n-2}=r_{n-1}=a_{n-1}+R_{n-1} \quad \text{and} \quad a_{n-2}+R_{n-2}\ge a_{n-1}+R_{n-1},$$

we get that

$$a_{n-2}+R_{n-2}=a_{n-1}+R_{n-1}=r_{n-2}=r_{n-1}.$$

Similarly, we get that

$$a_1+R_1=\cdots=a_n+R_n.$$

Next, if $a_1+R_1=\cdots=a_n+R_n$, since

$$a_n+R_n\le r_n\le\cdots\le r_1=a_1+R_1,$$

clearly, we have $r_n=\cdots=r_1$.

(2) When $n=1$, it clearly holds. When $n=2$, we have $f_2'(x)=2x-a_1-a_2$. Since

$$\begin{aligned}
&f_2(r_1)=R_1(r_1-a_2)-R_1R_2=R_1(a_1+R_1-a_2-R_2)\le 0,\\
&f_2(a_2+R_2)=(a_2+R_2-a_1)R_2-R_1R_2=(a_2+R_2-a_1-R_1)R_2\ge 0,\\
&f_2'(a_2+R_2)=2(a_2+R_2)-a_1-a_2=a_2+2R_2-a_1=a_2+R_2-a_1-R_1+R_1+R_2>0,
\end{aligned}$$

we obtain that $r_1\le r_2\le a_2+R_2$. When $n=3$, assume that $r_3<r_2$, then, we have $f_3(r_2)>0$. But

$$\begin{aligned}
f_3(r_2)&=R_1R_2(r_2-a_3)-R_1R_2R_3\\
&=R_1R_2(r_2-a_3-R_3)\\
&\le R_1R_2(a_2+R_2-(a_3+R_3))\le 0,
\end{aligned}$$

which is a contradiction. So we have $r_3\ge r_2$. It yields that

$$f_2(r_3)=(r_3-a_1)(r_3-a_2)-R_1R_2=\frac{R_1R_2R_3}{r_3-a_3}-R_1R_2=R_1R_2\left(\frac{R_3}{r_3-a_3}-1\right)\ge 0,$$

i.e., $r_3\le a_3+R_3$. Suppose that $r_{k-1}\le r_k\le a_k+R_k$ holds when $n=k$. Consider the case of $n=k+1$. Assume that $r_{k+1}<r_k$, then, we have $f_{k+1}(r_k)>0$. But

$$\begin{aligned}
f_{k+1}(r_k)&=R_1\cdots R_k(r_k-a_{k+1})-R_1\cdots R_kR_{k+1}\\
&=R_1\cdots R_k(r_k-a_{k+1}-R_{k+1})\\
&\le R_1\cdots R_k(a_k+R_k-a_{k+1}-R_{k+1})\le 0,
\end{aligned}$$

which is a contradiction. So we have $r_{k+1} \geq r_k$. It yields that

$$\begin{aligned} f_k(r_{k+1}) &= (r_{k+1}-a_1)\cdots(r_{k+1}-a_k) - R_1\cdots R_k \\ &= \frac{R_1\cdots R_k R_{k+1}}{r_{k+1}-a_{k+1}} - R_1\cdots R_k \\ &= R_1\cdots R_k\left(\frac{R_{k+1}}{r_{k+1}-a_{k+1}}-1\right) \geq 0, \end{aligned}$$

i.e., $r_{k+1} \leq a_{k+1}+R_{k+1}$. By mathematical induction, the result holds.

If $r_n = \cdots = r_1 = r$, then

$$\begin{aligned} f_n(r_n) &= f_n(r) = (r-a_1)\cdots(r-a_{n-1})(r-a_n) - R_1\cdots R_{n-1}R_n = 0, \\ f_{n-1}(r_{n-1}) &= f_{n-1}(r) = (r-a_1)\cdots(r-a_{n-1}) - R_1\cdots R_{n-1} = 0. \end{aligned}$$

It yields $r = a_n + R_n$. Since

$$a_n+R_n = r_n = r_{n-1} \leq a_{n-1}+R_{n-1} \quad \text{and} \quad a_{n-1}+R_{n-1} \leq a_n+R_n,$$

we get that

$$a_{n-1}+R_{n-1} = a_n+R_n = r_{n-1} = r_n.$$

Also,

$$a_{n-1}+R_{n-1} = r_{n-1} = r_{n-2} \leq a_{n-2}+R_{n-2} \quad \text{and} \quad a_{n-2}+R_{n-2} \geq a_{n-1}+R_{n-1},$$

we get that

$$a_{n-2}+R_{n-2} = a_{n-1}+R_{n-1} = r_{n-2} = r_{n-1}.$$

Similarly, we get that

$$a_1+R_1 = \cdots = a_n+R_n.$$

Next, if $a_1+R_1 = \cdots = a_n+R_n$, since

$$a_n+R_n \geq r_n \geq \cdots \geq r_1 = a_1+R_1,$$

clearly, we have $r_n = \cdots = r_1$. □

3 Inclusion sets of US-eigenvalues

For a complex symmetric tensor $\mathcal{S}$, let $\sigma(\mathcal{S})$ be the set of all US-eigenvalues of $\mathcal{S}$. From (b) of Theorem 1 in [1], we know $\sigma(\mathcal{S}) \subseteq \mathbb{R}$. In this section, we will always let

$$R_i(\mathcal{S}) = \sum_{\substack{i_2,\cdots,i_d=1 \\ (i_2,\cdots,i_d)\neq(i,\cdots,i)}}^{n} |s_{ii_2\cdots i_d}|.$$

Firstly, we give the Geršgorin-type inclusion set as following.

Theorem 3.1. *Let* $\mathcal{S}=(s_{i_1i_2\cdots i_d})\in\mathbb{C}^{n\times n\times\cdots n}$ *be a complex symmetric tensor. Then*

$$\sigma(\mathcal{S})\subseteq G(\mathcal{S})=\bigcup_{i=1}^{n}G_i(\mathcal{S}),$$

where

$$G_i(\mathcal{S})=\{z\in\mathbb{R}:|z|-|s_{ii\cdots i}|\le R_i(\mathcal{S})\}.$$

Proof. Let $\lambda\in\sigma(\mathcal{S})$ and $\boldsymbol{x}=(x_i)\in\mathbb{C}^n$ be the corresponding US-eigenvector. Let

$$|x_r|=\max\{|x_1|,|x_2|,\cdots,|x_n|\}.$$

Combining Eq. (1.2a) and Eq. (1.4a), we get

$$\lambda x_r^*=\sum_{\substack{i_2,\cdots,i_d=1\\(i_2,\cdots,i_d)\neq(r,\cdots,r)}}^{n} s^*_{ri_2\cdots i_d}x_{i_2}\cdots x_{i_d}+s^*_{rr\cdots r}x_r\cdots x_r.$$

Clearly,

$$|\lambda||x_r^*|\le\sum_{\substack{i_2,\cdots,i_d=1\\(i_2,\cdots,i_d)\neq(r,\cdots,r)}}^{n}\left|s^*_{ri_2\cdots i_d}\right||x_{i_2}|\cdots|x_{i_d}|+|s^*_{rr\cdots r}||x_r|^{d-1}.$$

Since $\|\boldsymbol{x}\|=1$, then $0<|x_r|\le 1$. Also,

$$\left|s^*_{i_1i_2\cdots i_d}\right|=|s_{i_1i_2\cdots i_d}|,\quad |x_i^*|=|x_i|,\quad i_k,i\in[n],\quad k\in[d],$$

it yields

$$|\lambda||x_r|\le\sum_{\substack{i_2,\cdots,i_d=1\\(i_2,\cdots,i_d)\neq(r,\cdots,r)}}^{n}|s_{ri_2\cdots i_d}||x_r|^{d-1}+|s_{rr\cdots r}||x_r|^{d-1},$$

then,

$$\begin{aligned}&|\lambda||x_r|^{d-1}\le R_r(\mathcal{S})|x_r|^{d-1}+|s_{rr\cdots r}||x_r|^{d-1},\\&|\lambda|-|s_{rr\cdots r}|\le R_r(\mathcal{S}).\end{aligned}$$

Similarly, combining Eq. (1.2b) and Eq. (1.4b), we also get

$$|\lambda|-|s_{rr\cdots r}|\le R_r(\mathcal{S}).$$

Therefore, the result holds. □

Secondly, we give the Brauer-type inclusion set as following.

Theorem 3.2. *Let* $\mathcal{S}=(s_{i_1i_2\cdots i_d})\in\mathbb{C}^{n\times n\times\cdots n}$ *be a complex symmetric tensor. Then*

$$\sigma(\mathcal{S})\subseteq B(\mathcal{S})=\left(\bigcup_{i,j=1,i\neq j}^{n}B_{i,j}(\mathcal{S})\right)\cup F(\mathcal{S}),$$

where

$$\begin{aligned}&B_{i,j}(\mathcal{S})=\{z\in\mathbb{R}:(|z|-|s_{ii\cdots i}|)(|z|-|s_{jj\cdots j}|)\leq R_i(\mathcal{S})R_j(\mathcal{S})\},\\&F(\mathcal{S})=\bigcup_{i=1}^{d}\{z\in\mathbb{R}:|z|\leq|s_{ii\cdots i}|\}.\end{aligned}$$

Proof. Let $\lambda\in\sigma(\mathcal{S})$ and $\boldsymbol{x}=(x_i)\in\mathbb{C}^n$ be the corresponding US-eigenvector. Let

$$|x_p|=\max\{|x_i|:i\in[n]\},\quad |x_q|=\max\{|x_i|:i\in[n],\ i\neq p\}.$$

Combining Eq. (1.2a) and Eq. (1.4a), we get

$$\lambda x_p^*=\sum_{\substack{i_2,\cdots,i_d=1\\(i_2,\cdots,i_d)\neq(p,\cdots,p)}}^{n}s^*_{pi_2\cdots i_d}x_{i_2}\cdots x_{i_d}+s^*_{pp\cdots p}x_p\cdots x_p.$$

Clearly,

$$|\lambda||x_p^*|\leq\sum_{\substack{i_2,\cdots,i_d=1\\(i_2,\cdots,i_d)\neq(p,\cdots,p)}}^{n}\left|s^*_{pi_2\cdots i_d}\right||x_{i_2}|\cdots|x_{i_d}|+\left|s^*_{pp\cdots p}\right||x_p|^{d-1}.$$

From $\|\boldsymbol{x}\|=1$, we have $0<|x_p|\leq 1$. Also,

$$\left|s^*_{i_1i_2\cdots i_d}\right|=|s_{i_1i_2\cdots i_d}|,\quad |x_i^*|=|x_i|,\quad i_k,i\in[n],\quad k\in[d],$$

we get

$$|\lambda||x_p|\leq\sum_{\substack{i_2,\cdots,i_d=1\\(i_2,\cdots,i_d)\neq(p,\cdots,p)}}^{n}|s_{pi_2\cdots i_d}||x_p|^{d-2}|x_q|+|s_{pp\cdots p}||x_p|^{d-1},$$

then,

$$\begin{aligned}&|\lambda||x_p|^{d-1}\leq R_p(\mathcal{S})|x_p|^{d-2}|x_q|+|s_{pp\cdots p}||x_p|^{d-1},\\&(|\lambda|-|s_{pp\cdots p}|)|x_p|\leq R_p(\mathcal{S})|x_q|.\end{aligned}\tag{3.1}$$

When $x_q=0$, then from Eq. (3.1), we have $(|\lambda|-|s_{pp\cdots p}|)|x_p|\leq 0$, i.e., $|\lambda|\leq|s_{pp\cdots p}|$. Clearly, $\lambda\in B(\mathcal{S})$. When $x_q\neq 0$, then we have

$$\lambda x_q^*=\sum_{\substack{i_2,\cdots,i_d=1\\(i_2,\cdots,i_d)\neq(q,\cdots,q)}}^{n} s^*_{qi_2\cdots i_d}x_{i_2}\cdots x_{i_d}+s^*_{qq\cdots q}x_q\cdots x_q.$$

Clearly,

$$|\lambda|\left|x_q^*\right|\leq\sum_{\substack{i_2,\cdots,i_d=1\\(i_2,\cdots,i_d)\neq(q,\cdots,q)}}^{n}\left|s^*_{qi_2\cdots i_d}\right||x_{i_2}|\cdots|x_{i_d}|+\left|s^*_{qq\cdots q}\right||x_q|^{d-1},$$

then,

$$\begin{aligned}&|\lambda||x_q||x_p|^{d-2}\leq\sum_{\substack{i_2,\cdots,i_d=1\\(i_2,\cdots,i_d)\neq(q,\cdots,q)}}^{n}|s_{qi_2\cdots i_d}||x_p|^{d-1}+|s_{qq\cdots q}||x_q||x_p|^{d-2},\\&(|\lambda|-|s_{qq\cdots q}|)|x_q|\leq R_q(\mathcal{S})|x_p|.\end{aligned}\tag{3.2}$$

Multiply Eq. (3.1) and Eq. (3.2), we conclude that

$$(|\lambda|-|s_{pp\cdots p}|)(|\lambda|-|s_{qq\cdots q}|)\leq R_p(\mathcal{S})R_q(\mathcal{S}).$$

Similarly, combining Eq. (1.2b) and Eq. (1.4b), it yields that

$$(|\lambda|-|s_{pp\cdots p}|)(|\lambda|-|s_{qq\cdots q}|)\leq R_p(\mathcal{S})R_q(\mathcal{S}).$$

Therefore, the result holds. $\square$

Next, we give the Brualdi-type inclusion set as following.

Theorem 3.3. *Let $\mathcal{S}=(s_{i_1i_2\cdots i_d})\in\mathbb{C}^{n\times n\times\cdots n}$ be a complex symmetric tensor. If $\Gamma_{\mathcal{S}}$ is weakly connected, then*

$$\sigma(\mathcal{S})\subseteq D(\mathcal{S})=\left(\bigcup_{\gamma\in C(\mathcal{S})}D_\gamma(\mathcal{S})\right)\cup F(\mathcal{S}),$$

where

$$D_\gamma(\mathcal{S})=\left\{z\in\mathbb{R}:\prod_{i\in\gamma}(|z|-|s_{ii\cdots i}|)\leq\prod_{i\in\gamma}R_i(\mathcal{S})\right\},$$

$C(\mathcal{S})$ denotes the set of all circuits in $\Gamma_{\mathcal{S}}$, and $F(\mathcal{S})$ is the same as in Theorem 3.2.

Proof. Let $\lambda\in\sigma(\mathcal{S})$. Since $\Gamma_{\mathcal{S}}$ is weakly connected, then $\lambda\in D(\mathcal{S})$ if $|\lambda|\le|s_{ii\cdots i}|$ for some $i\in[n]$. Suppose that $|\lambda|>|s_{ii\cdots i}|$ for all $i\in[n]$. Let $\boldsymbol{x}=(x_i)\in\mathbb{C}^n$ be an US-eigenvector corresponding to λ and Γ_0 be the subgraph of $\Gamma_{\mathcal{A}}$ induced by those vertices v_i for which $x_i\neq 0$. Combining Eq. (1.2a) and Eq. (1.4a), we get

$$\lambda x_i^* = \sum_{\substack{i_2,\cdots,i_d=1\\(i_2,\cdots,i_d)\neq(i,\cdots,i)}}^{n} s^*_{ii_2\cdots i_d}x_{i_2}\cdots x_{i_d}+s^*_{ii\cdots i}x_i\cdots x_i.$$

Clearly,

$$|\lambda||x_i^*|\le \sum_{\substack{i_2,\cdots,i_d=1\\(i_2,\cdots,i_d)\neq(i,\cdots,i)}}^{n} \left|s^*_{ii_2\cdots i_d}\right||x_{i_2}\cdots x_{i_d}|+|s^*_{ii\cdots i}|\left|x_i^{d-1}\right|.$$

Since $\|\boldsymbol{x}\|=1$, we have $0<|x_i|\le 1$, $i\in[n]$. Also,

$$\left|s^*_{i_1i_2\cdots i_d}\right|=|s_{i_1i_2\cdots i_d}|,\quad |x_i^*|=|x_i|,\quad i_k,i\in[n],\quad k\in[d],$$

it yields

$$\begin{aligned}|\lambda|\left|x_i^{d-1}\right|\le|\lambda||x_i|&\le \sum_{\substack{i_2,\cdots,i_d=1\\(i_2,\cdots,i_d)\neq(i,\cdots,i)}}^{n} |s_{ii_2\cdots i_d}||x_{i_2}\cdots x_{i_d}|+|s_{ii\cdots i}|\left|x_i^{d-1}\right|,\\ (|\lambda|-|s_{ii\cdots i}|)\left|x_i^{d-1}\right|&\le \sum_{\substack{i_2,\cdots,i_d=1\\(i_2,\cdots,i_d)\neq(i,\cdots,i)}}^{n} |s_{ii_2\cdots i_d}||x_{i_2}\cdots x_{i_d}|.\end{aligned}$$

Since $|\lambda|>|s_{ii\cdots i}|$, by the above inequality, we know that $\Gamma_0^+(v_i)$ is nonempty for each vertex v_i in Γ_0. Define a pre-order: $v_i\le v_j$ on the vertex set of Γ_0 if and only if $|x_i|\le|x_j|$. According to Lemma 2.1, there exits a circuit $\gamma=\{v_1,\cdots,v_p,v_{p+1}=v_1\}$ in Γ_0 and it satisfies that $\left|x_{i_{j+1}}\right|\ge|x_k|$ for each $v_k\in\Gamma_0^+(v_j)$, $j\in[p]$. Then we conclude that for each $j\in[p]$,

$$\left(|\lambda|-\left|s_{i_ji_j\cdots i_j}\right|\right)\left|x_{i_j}^{d-1}\right|\le \sum_{\substack{i_2,\cdots,i_d=1\\(i_2,\cdots,i_d)\neq(i_j,\cdots,i_j)}}^{n} \left|s_{i_ji_2\cdots i_d}\right|\left|x_{i_{j+1}}^{d-1}\right|.$$

Hence,

$$\prod_{j=1}^{p}\left(|\lambda|-\left|s_{i_ji_j\cdots i_j}\right|\right)\prod_{j=1}^{p}\left|x_{i_j}^{d-1}\right|\le\prod_{j=1}^{p}R_{i_j}(\mathcal{S})\prod_{j=1}^{p}\left|x_{i_{j+1}}^{d-1}\right|.$$

Since $v_{p+1}=v_1$, $x_j\neq 0$, $j\in[p]$,

$$\prod_{j=1}^{p}\left(|\lambda|-\left|s_{i_ji_j\cdots i_j}\right|\right)\leq\prod_{j=1}^{p}R_{i_j}(\mathcal{S}),$$

that is

$$\prod_{i\in\gamma}(|\lambda|-|s_{ii\cdots i}|)\leq\prod_{i\in\gamma}R_i(\mathcal{S}).$$

Similarly, combining Eq. (1.2b) and Eq. (1.4b), we also get

$$\prod_{i\in\gamma}(|\lambda|-|s_{ii\cdots i}|)\leq\prod_{i\in\gamma}R_i(\mathcal{S}).$$

Therefore, the result holds. □

Now, the relation among the three inclusion sets is shown as following.

Theorem 3.4. *For*

$$D(\mathcal{S})\subseteq B(\mathcal{S})\subseteq G(\mathcal{S}),$$

where $G(\mathcal{S})$, $B(\mathcal{S})$, $D(\mathcal{S})$ *are the same as in Theorem* 3.1*, Theorem* 3.2 *and Theorem* 3.3*, respectively.*

Proof. (1) Let's do a sort: $|s_{k_1\cdots k_1}|+R_{k_1}(\mathcal{S})\geq\cdots\geq\left|s_{k_p\cdots k_p}\right|+R_{k_p}(\mathcal{S})$ with $R_{k_i}(\mathcal{S})\neq 0$, $i\in[p]$, and another sort: $|s_{l_1\cdots l_1}|\geq\cdots\geq\left|s_{l_q\cdots l_q}\right|$ with $R_{l_i}(\mathcal{S})=0$, $i\in[q]$, $p+q=n$. For all $i\in[q]$, $j\in[n]$, $l_i\neq j$,

$$\{z\in\mathbb{R}:(|z|-|s_{l_i\cdots l_i}|)(|z|-|s_{j\cdots j}|)\leq 0\}\subseteq E\subseteq G(\mathcal{S}).$$

For all $i,j\in[p]$, $i<j$, by Lemma 2.2,

$$\begin{aligned}&\left\{z\in\mathbb{R}:(|z|-|s_{k_i\cdots k_i}|)\left(|z|-\left|s_{k_j\cdots k_j}\right|\right)\leq R_{k_i}(\mathcal{S})R_{k_j}(\mathcal{S})\right\}\\ \subseteq&\{z\in\mathbb{R}:|z|-|s_{k_i\cdots k_i}|\leq R_{k_i}(\mathcal{S})\}.\end{aligned}$$

Hence, $B(\mathcal{S})\subseteq G(\mathcal{S})$.

(2) Assume that there exits an $\lambda\in\sigma(\mathcal{S})$ with $\lambda\notin B(\mathcal{S})$. Clearly, $\lambda\notin F(\mathcal{S})$, where $F(\mathcal{S})$ is denoted in Theorem 3.2. For a circuit $\gamma_0\in C(\mathcal{S})$, there are vertices $v_{t_1},\cdots,v_{t_m}$ in γ_0. According to Theorem 3.2, we have

$$\begin{aligned}&(|\lambda|-|s_{t_i\cdots t_i}|)\left(|\lambda|-\left|s_{t_j\cdots t_j}\right|\right)\geq R_{t_i}(\mathcal{S})R_{t_j}(\mathcal{S}),\\ &|\lambda|-|s_{t_i\cdots t_i}|>0,\end{aligned}\tag{3.3}$$

where $i,j\in[m]$, $i\neq j$. If m is even, we get that

$$\prod_{i\in\gamma_0}(|\lambda|-|s_{i\cdots i}|)\geq\prod_{i\in\gamma_0}R_i(\mathcal{S}),$$

which shows $\lambda\notin D(\mathcal{S})$. If m is odd, since $\Gamma_{\mathcal{S}}$ is weakly connected, $R_i(\mathcal{S})>0$, $i\in\gamma_0$. So that the Eq. (3.3) can become the following form:

$$\frac{|\lambda|-|s_{t_i\cdots t_i}|}{R_{t_i}(\mathcal{S})}\frac{|\lambda|-|s_{t_j\cdots t_j}|}{R_{t_j}(\mathcal{S})}\geq 1.$$

Then there exits $u\in[m]$ with

$$\frac{|\lambda|-|s_{t_u\cdots t_u}|}{R_{t_u}(\mathcal{S})}\geq 1,$$

i.e.,

$$|\lambda|-|s_{t_u\cdots t_u}|\geq R_{t_u}(\mathcal{S}).$$

Then we get that

$$\left(\prod_{i\in[m]\backslash\{u\}}(|\lambda|-|s_{t_i\cdots t_i}|)\right)(|\lambda|-|s_{t_u\cdots t_u}|)\geq\left(\prod_{i\in[m]\backslash\{u\}}R_{t_i}(\mathcal{S})\right)R_{t_u}(\mathcal{S}),$$

i.e.,

$$\prod_{i\in\gamma_0}(|\lambda|-|s_{i\cdots i}|)\geq\prod_{i\in\gamma_0}R_i(\mathcal{S}),$$

which shows $\lambda\notin D(\mathcal{S})$. Hence, $D(\mathcal{S})\subseteq B(\mathcal{S})$. Therefore, the results hold. □

Finally, we give an example.

Example 3.1. Let $\mathcal{S}\in\mathbb{C}^{4\times4\times4\times4}$ be a complex symmetric tensor, where $(\mathcal{S})_{1111}=1$, $(\mathcal{S})_{2222}=2+\sqrt{5}i$, $(\mathcal{S})_{3333}=-6$, $(\mathcal{S})_{4444}=5i$, $(\mathcal{S})_{1112}=(\mathcal{S})_{1121}=(\mathcal{S})_{1211}=(\mathcal{S})_{2111}=\frac{i}{2}$, $(\mathcal{S})_{1113}=(\mathcal{S})_{1131}=(\mathcal{S})_{1311}=(\mathcal{S})_{3111}=-\frac{i}{2}$, $(\mathcal{S})_{2224}=(\mathcal{S})_{2242}=(\mathcal{S})_{2422}=(\mathcal{S})_{4222}=\frac{i}{2}$, and other elements are 0. After calculation, we obtain the following US-eigenvalue inclusion sets:

the Geršgorin-type inclusion set is

$$G(\mathcal{S})=\{\lambda\in\mathbb{C}:|\lambda|\leq 6.5\},$$

the Brauer-type inclusion set is

$$B(\mathcal{S})=\left\{\lambda\in\mathbb{C}:|\lambda|\leq\frac{9+\sqrt{13}}{2}\approx 6.3028\right\},$$

the Brualdi-type inclusion set is

$$D(\mathcal{S})=\left\{\lambda\in\mathbb{C}:|\lambda|\le\frac{7+\sqrt{31}}{2}\approx 6.2839\right\}.$$

4 Inclusion sets of U-eigenvalues

For a general complex tensor $\mathcal{A}$, let $\sigma(\mathcal{A})$ be the set of all U-eigenvalues of $\mathcal{A}$. From (b) of Theorem 1 in [1], we know $\sigma(\mathcal{A})\subseteq\mathbb{R}$. In this section, we will always let

(1) $i^{[n]}=\begin{cases} i, & i\in[n],\\ n, & i\notin[n],\end{cases}$

(2) $R_{k,i}(\mathcal{A})=\sum\limits_{\substack{i_1,\cdots,i_{k-1},i_{k+1},\cdots,i_d=1\\ (i_1,\cdots,i_{k-1},i_{k+1},\cdots,i_d)\neq\\ \left(i^{[n_1]},\cdots,i^{[n_{k-1}]},i^{[n_{k+1}]}\cdots,i^{[n_d]}\right)}}^{n_1,\cdots,n_{k-1},n_k,\cdots,n_t}\left|a_{i_1\cdots i_{k-1}ii_{k+1}\cdots i_d}\right|.$

Firstly, we give the Geršgorin-type inclusion set as following.

Theorem 4.1. *Let $\mathcal{A}=(a_{i_1i_2\cdots i_d})$ be a d-order $(n_1\times n_2\times\cdots\times n_d)$-dimensional complex tensor. Then*

$$\sigma(\mathcal{A})\subseteq G(\mathcal{A})=\bigcup_{k=1}^{d}G_k(\mathcal{A}),$$

where

$$G_k(\mathcal{A})=\bigcup_{i=1}^{n_k}\left\{z\in\mathbb{R}:|z|-\left|a_{i^{[n_1]}\cdots i^{[n_{k-1}]}ii^{[n_{k+1}]}\cdots i^{[n_d]}}\right|\le R_{k,i}(\mathcal{A})\right\}.$$

Proof. Let $\lambda\in\sigma(\mathcal{A})$, and $\otimes_{i=1}^{d}\boldsymbol{x}^{(i)}\left(\boldsymbol{x}^{(i)}\in\mathbb{C}^{n_i}\right)$ be the corresponding U-eigenvector pairs. Let

$$\left|x_m^{(s)}\right|=\max\left\{\left|x_{i_k}^{(k)}\right|:i_k\in[n_k],\ k\in[d]\right\}.$$

Combining Eq. (1.1a) and Eq. (1.3a), we get

$$\begin{aligned}\lambda x_m^{(s)*}=&a^*_{m^{[n_1]}\cdots m^{[n_{s-1}]}mm^{[n_{s+1}]}\cdots m^{[n_d]}}x^{(1)}_{m^{[n_1]}}\cdots x^{(s-1)}_{m^{[n_{s-1}]}}x^{(s+1)}_{m^{[n_{s+1}]}}\cdots x^{(d)}_{m^{[n_d]}}\\ &+\sum_{\substack{i_1,\cdots,i_{s-1},i_{s+1},\cdots,i_d=1\\ (i_1,\cdots,i_{s-1},i_{s+1},\cdots,i_d)\neq\\ \left(m^{[n_1]},\cdots,m^{[n_{s-1}]},m^{[n_{s+1}]}\cdots,m^{[n_d]}\right)}}^{n_1,\cdots,n_{s-1},n_{s+1},\cdots,n_d}a^*_{i_1\cdots i_{s-1}mi_{s+1}\cdots i_d}x^{(1)}_{i_1}\cdots x^{(s-1)}_{i_{s-1}}x^{(s+1)}_{i_{s+1}}\cdots x^{(d)}_{i_d}.\end{aligned}$$

Clearly,

$$
\begin{aligned}
|\lambda|\left|x_m^{(s)*}\right| \leq & \left|a^*_{m^{[n_1]}\cdots m^{[n_{s-1}]}mm^{[n_{s+1}]}\cdots m^{[n_d]}}\right|\left|x^{(1)}_{m^{[n_1]}}\cdots x^{(s-1)}_{m^{[n_{s-1}]}}x^{(s+1)}_{m^{[n_{s+1}]}}\cdots x^{(d)}_{m^{[n_d]}}\right| \\
& + \sum_{\substack{i_1,\cdots,i_{s-1},i_{s+1},\cdots,i_d=1 \\ (i_1,\cdots,i_{s-1},i_{s+1},\cdots,i_d)\neq \\ \left(m^{[n_1]},\cdots,m^{[n_{s-1}]},m^{[n_{s+1}]}\cdots,m^{[n_d]}\right)}}^{n_1,\cdots,n_{s-1},n_{s+1},\cdots,n_d} \left|a^*_{i_1\cdots i_{s-1}mi_{s+1}\cdots i_d}\right|\left|x^{(1)}_{i_1}\cdots x^{(s-1)}_{i_{s-1}}x^{(s+1)}_{i_{s+1}}\cdots x^{(d)}_{i_d}\right|.
\end{aligned}
$$

Since $\left\|\boldsymbol{x}^{(k)}\right\|=1$, $k\in[d]$, we have $0<|x_m^{(s)}|\leq 1$. Also,

$$
\left|a^*_{i_1i_2\cdots i_d}\right|=|a_{i_1i_2\cdots i_d}|, \quad \left|x^{(k)*}_{i_k}\right|=\left|x^{(k)}_{i_k}\right|, \quad i_k\in[n_k], \quad k\in[d],
$$

we get

$$
\begin{aligned}
|\lambda|\left|x_m^{(s)*}\right| \leq & \left|a_{m^{[n_1]}\cdots m^{[n_{s-1}]}mm^{[n_{s+1}]}\cdots m^{[n_d]}}\right|\left|x^{(1)}_{m^{[n_1]}}\cdots x^{(s-1)}_{m^{[n_{s-1}]}}x^{(s+1)}_{m^{[n_{s+1}]}}\cdots x^{(d)}_{m^{[n_d]}}\right| \\
& + \sum_{\substack{i_1,\cdots,i_{s-1},i_{s+1},\cdots,i_d=1 \\ (i_1,\cdots,i_{s-1},i_{s+1},\cdots,i_d)\neq \\ \left(m^{[n_1]},\cdots,m^{[n_{s-1}]},m^{[n_{s+1}]}\cdots,m^{[n_d]}\right)}}^{n_1,\cdots,n_{s-1},n_{s+1},\cdots,n_d} \left|a_{i_1\cdots i_{s-1}mi_{s+1}\cdots i_d}\right|\left|x^{(1)}_{i_1}\cdots x^{(s-1)}_{i_{s-1}}x^{(s+1)}_{i_{s+1}}\cdots x^{(d)}_{i_d}\right|,
\end{aligned}
$$

then,

$$
\begin{aligned}
&|\lambda|\left|x_m^{(s)}\right|^{d-1} \leq \left|a_{m^{[n_1]}\cdots m^{[n_{s-1}]}mm^{[n_{s+1}]}\cdots m^{[n_d]}}\right|\left|x_m^{(s)}\right|^{d-1}+R_{s,m}(\mathcal{A})\left|x_m^{(s)}\right|^{d-1}, \\
&|\lambda|-\left|a_{m^{[n_1]}\cdots m^{[n_{s-1}]}mm^{[n_{s+1}]}\cdots m^{[n_d]}}\right| \leq R_{s,m}(\mathcal{A}).
\end{aligned}
$$

Similarly, combining Eq. (1.1b) and Eq. (1.3b), it yields that

$$
|\lambda|-\left|a_{m^{[n_1]}\cdots m^{[n_{s-1}]}mm^{[n_{s+1}]}\cdots m^{[n_d]}}\right| \leq R_{s,m}(\mathcal{A}).
$$

Therefore, the result holds. □

Next, we give the Brauer-type inclusion set as following.

Theorem 4.2. *Let $\mathcal{A}=(a_{i_1i_2\cdots i_d})$ be a d-order $(n_1\times n_2\times\cdots\times n_d)$-dimensional complex tensor. Then*

$$
\sigma(\mathcal{A})\subseteq B(\mathcal{A})=\left(\bigcup_{\substack{p,q=1\\p\neq q}}^{d} B_{p,q}(\mathcal{A})\right)\cup F(\mathcal{A}),
$$

where

$$B_{p,q}(\mathcal{A})=\bigcup_{\substack{i,j=1\\ i\neq j}}^{n_p,n_q}\Big\{z\in\mathbb{R}:\Big(|z|-\Big|a_{i^{[n_1]}\cdots i^{[n_{p-1}]}ii^{[n_{p+1}]}\cdots i^{[n_d]}}\Big|\Big)\Big(|z|-\Big|a_{j^{[n_1]}\cdots j^{[n_{q-1}]}jj^{[n_{q+1}]}\cdots j^{[n_d]}}\Big|\Big)$$
$$\leq R_{p,i}(\mathcal{A})R_{q,j}(\mathcal{A})\Big\},$$
$$F(\mathcal{A})=\bigcup_{i=1}^{\max\{n_1,\cdots,n_d\}}\Big\{z\in\mathbb{R}:|z|\leq\Big|a_{i^{[n_1]}\cdots i^{[n_d]}}\Big|\Big\}.$$

Proof. Let $\lambda\in\sigma(\mathcal{A})$, and $\otimes_{i=1}^{d}\boldsymbol{x}^{(i)}\left(\boldsymbol{x}^{(i)}\in\mathbb{C}^{n_i}\right)$ be the corresponding U-eigenvector pairs. Let

$$\left|x_m^{(s)}\right|=\max\Big\{\Big|x_{i_k}^{(k)}\Big|:i_k\in[n_k],\ k\in[d]\Big\},$$
$$\left|x_l^{(r)}\right|=\max\Big\{\Big|x_{i_k}^{(k)}\Big|:i_k\in[n_k]\backslash\{m\},\ k\in[d]\backslash\{s\}\Big\}.$$

Combining Eq. (1.1a) and Eq. (1.3a), we get

$$\lambda x_m^{(s)*}=a^*_{m^{[n_1]}\cdots m^{[n_{s-1}]}mm^{[n_{s+1}]}\cdots m^{[n_d]}}x^{(1)}_{l^{[n_1]}}\cdots x^{(s-1)}_{m^{[n_{s-1}]}}x^{(s+1)}_{m^{[n_{s+1}]}}\cdots x^{(d)}_{m^{[n_d]}}$$
$$+\sum_{\substack{i_1,\cdots,i_{s-1},i_{s+1},\cdots,i_d=1\\ (i_1,\cdots,i_{s-1},i_{s+1},\cdots,i_d)\neq\\ \left(m^{[n_1]},\cdots,m^{[n_{s-1}]},m^{[n_{s+1}]}\cdots,m^{[n_d]}\right)}}^{n_1,\cdots,n_{s-1},n_{s+1},\cdots,n_d}a^*_{i_1\cdots i_{s-1}mi_{s+1}\cdots i_d}x^{(1)}_{i_1}\cdots x^{(s-1)}_{i_{s-1}}x^{(s+1)}_{i_{s+1}}\cdots x^{(d)}_{i_d}.$$

Clearly,

$$|\lambda|\left|x_m^{(s)*}\right|\leq\Big|a^*_{m^{[n_1]}\cdots m^{[n_{s-1}]}mm^{[n_{s+1}]}\cdots m^{[n_d]}}\Big|\Big|x^{(1)}_{m^{[n_1]}}\cdots x^{(s-1)}_{m^{[n_{s-1}]}}x^{(s+1)}_{m^{[n_{s+1}]}}\cdots x^{(d)}_{m^{[n_d]}}\Big|$$
$$+\sum_{\substack{i_1,\cdots,i_{s-1},i_{s+1},\cdots,i_d=1\\ (i_1,\cdots,i_{s-1},i_{s+1},\cdots,i_d)\neq\\ \left(m^{[n_1]},\cdots,m^{[n_{s-1}]},m^{[n_{s+1}]}\cdots,m^{[n_d]}\right)}}^{n_1,\cdots,n_{s-1},n_{s+1},\cdots,n_d}\Big|a^*_{i_1\cdots i_{s-1}mi_{s+1}\cdots i_d}\Big|\Big|x^{(1)}_{i_1}\cdots x^{(s-1)}_{i_{s-1}}x^{(s+1)}_{i_{s+1}}\cdots x^{(d)}_{i_d}\Big|.$$

Since $\left\|\boldsymbol{x}^{(k)}\right\|=1$, $k\in[d]$, then $0<|x_m^{(s)}|\leq 1$. Also

$$\left|a^*_{i_1i_2\cdots i_d}\right|=|a_{i_1i_2\cdots i_d}|,\quad\left|x_{i_k}^{(k)*}\right|=\left|x_{i_k}^{(k)}\right|,\quad i_k\in[n_k],\quad k\in[d],$$

we get

$$\begin{aligned}|\lambda|\left|x_m^{(s)}\right|\left|x_l^{(r)}\right|^{d-2}\leq&\left|a_{m^{[n_1]}\cdots m^{[n_{s-1}]}mm^{[n_{s+1}]}\cdots m^{[n_d]}}\right|\left|x_m^{(s)}\right|\left|x_l^{(r)}\right|^{d-2}\\&+\sum_{\substack{i_1,\cdots,i_{s-1},i_{s+1},\cdots,i_d=1\\(i_1,\cdots,i_{s-1},i_{s+1},\cdots,i_d)\neq\\\left(m^{[n_1]},\cdots,m^{[n_{s-1}]},m^{[n_{s+1}]}\cdots,m^{[n_d]}\right)}}^{n_1,\cdots,n_{s-1},n_{s+1},\cdots,n_d}\left|a_{i_1\cdots i_{s-1}mi_{s+1}\cdots i_d}\right|\left|x_l^{(r)}\right|^{d-1},\end{aligned}$$

then,

$$\left(|\lambda|-\left|a_{m^{[n_1]}\cdots m^{[n_{s-1}]}mm^{[n_{s+1}]}\cdots m^{[n_d]}}\right|\right)\left|x_m^{(s)}\right|\leq R_{s,m}(\mathcal{A})\left|x_l^{(r)}\right|. \tag{4.1}$$

Meanwhile, we get

$$\begin{aligned}\lambda x_l^{(r)*}=&a^*_{l^{[n_1]}\cdots l^{[n_{r-1}]}ll^{[n_{r+1}]}\cdots l^{[n_d]}}x^{(1)}_{l^{[n_1]}}\cdots x^{(r-1)}_{l^{[n_{r-1}]}}x^{(r+1)}_{l^{[n_{r+1}]}}\cdots x^{(d)}_{l^{[n_d]}}\\&+\sum_{\substack{i_1,\cdots,i_{r-1},i_{r+1},\cdots,i_d=1\\(i_1,\cdots,i_{r-1},i_{r+1},\cdots,i_d)\neq\\\left(l^{[n_1]},\cdots,l^{[n_{r-1}]},l^{[n_{r+1}]}\cdots,l^{[n_d]}\right)}}^{n_1,\cdots,n_{r-1},n_{r+1},\cdots,n_d}a^*_{i_1\cdots i_{r-1}li_{r+1}\cdots i_d}x^{(1)}_{i_1}\cdots x^{(r-1)}_{i_{r-1}}x^{(r+1)}_{i_{r+1}}\cdots x^{(d)}_{i_d}.\end{aligned}$$

Clearly,

$$\begin{aligned}|\lambda|\left|x_l^{(r)*}\right|\leq&\left|a^*_{l^{[n_1]}\cdots l^{[n_{r-1}]}ll^{[n_{r+1}]}\cdots l^{[n_d]}}\right|\left|x^{(1)}_{l^{[n_1]}}\cdots x^{(r-1)}_{l^{[n_{r-1}]}}x^{(r+1)}_{l^{[n_{r+1}]}}\cdots x^{(d)}_{l^{[n_d]}}\right|\\&+\sum_{\substack{i_1,\cdots,i_{r-1},i_{r+1},\cdots,i_d=1\\(i_1,\cdots,i_{r-1},i_{r+1},\cdots,i_d)\neq\\\left(l^{[n_1]},\cdots,l^{[n_{r-1}]},l^{[n_{r+1}]}\cdots,l^{[n_d]}\right)}}^{n_1,\cdots,n_{r-1},n_{r+1},\cdots,n_d}\left|a^*_{i_1\cdots i_{r-1}li_{r+1}\cdots i_d}\right|\left|x^{(1)}_{i_1}\cdots x^{(r-1)}_{i_{r-1}}x^{(r+1)}_{i_{r+1}}\cdots x^{(d)}_{i_d}\right|,\end{aligned}$$

then,

$$\begin{aligned}|\lambda|\left|x_l^{(r)}\right|\left|x_m^{(s)}\right|^{d-2}\leq&\left|a_{l^{[n_1]}\cdots l^{[n_{r-1}]}ll^{[n_{r+1}]}\cdots l^{[n_d]}}\right|\left|x_l^{(s)}\right|\left|x_m^{(s)}\right|^{d-2}\\&+\sum_{\substack{i_1,\cdots,i_{r-1},i_{r+1},\cdots,i_d=1\\(i_1,\cdots,i_{r-1},i_{r+1},\cdots,i_d)\neq\\\left(l^{[n_1]},\cdots,l^{[n_{r-1}]},l^{[n_{r+1}]}\cdots,l^{[n_d]}\right)}}^{n_1,\cdots,n_{r-1},n_{r+1},\cdots,n_d}\left|a_{i_1\cdots i_{r-1}li_{r+1}\cdots i_d}\right|\left|x_m^{(s)}\right|^{d-1},\end{aligned}$$

and

$$\left(|\lambda|-\left|a_{l^{[n_1]}\cdots l^{[n_{r-1}]}ll^{[n_{r+1}]}\cdots l^{[n_d]}}\right|\right)\left|x_l^{(r)}\right|\leq R_{r,l}(\mathcal{A})\left|x_m^{(s)}\right|. \tag{4.2}$$

If $x_l^{(r)}=0$, from Eq. (4.1), we have

$$\left(|\lambda|-\left|a_{m^{[n_1]}\cdots m^{[n_{s-1}]}mm^{[n_{s+1}]}\cdots m^{[n_d]}}\right|\right)\left|x_m^{(s)}\right|\leq 0,$$

i.e.,

$$|\lambda|\leq\left|a_{m^{[n_1]}\cdots m^{[n_{s-1}]}mm^{[n_{s+1}]}\cdots m^{[n_d]}}\right|.$$

It yields $\lambda\in B(\mathcal{A})$. If $x_l^{(r)}\neq 0$, multiple Eq. (4.1) and Eq. (4.2), we conclude that

$$\begin{aligned}&\left(|\lambda|-\left|a_{m^{[n_1]}\cdots m^{[n_{s-1}]}mm^{[n_{s+1}]}\cdots m^{[n_d]}}\right|\right)\left(|\lambda|-\left|a_{l^{[n_1]}\cdots l^{[n_{r-1}]}ll^{[n_{r+1}]}\cdots l^{[n_d]}}\right|\right)\\ \leq&R_{m,s}(\mathcal{A})R_{r,l}(\mathcal{A}).\end{aligned}$$

Similarly, combining Eq. (1.1b) and Eq. (1.3b), we have

$$\begin{aligned}&\left(|\lambda|-\left|a_{m^{[n_1]}\cdots m^{[n_{s-1}]}mm^{[n_{s+1}]}\cdots m^{[n_d]}}\right|\right)\left(|\lambda|-\left|a_{l^{[n_1]}\cdots l^{[n_{r-1}]}ll^{[n_{r+1}]}\cdots l^{[n_d]}}\right|\right)\\ \leq&R_{m,s}(\mathcal{A})R_{r,l}(\mathcal{A}).\end{aligned}$$

Therefore, the result holds. □

Now, the relation between the two inclusion sets is shown as following.

Theorem 4.3. *For*

$$B(\mathcal{A})\subseteq G(\mathcal{A}),$$

where $G(\mathcal{A})$, $B(\mathcal{A})$ *are the same as in Theorem* 4.1 *and Theorem* 4.2, *respectively.*

Proof. For a tensor $\mathcal{A}$, we have the following numbers:

$$\begin{aligned}&\left|a_{1^{[n_1]}\cdots 1^{[n_d]}}\right|+R_{1,1}(\mathcal{A}),\left|a_{2^{[n_1]}\cdots 2^{[n_d]}}\right|+R_{1,2}(\mathcal{A}),\cdots,\left|a_{n_1^{[n_1]}\cdots n_1^{[n_d]}}\right|+R_{1,n_1}(\mathcal{A}),\\ &\left|a_{1^{[n_1]}\cdots 1^{[n_d]}}\right|+R_{2,1}(\mathcal{A}),\left|a_{2^{[n_1]}\cdots 2^{[n_d]}}\right|+R_{2,2}(\mathcal{A}),\cdots,\left|a_{n_2^{[n_1]}\cdots n_2^{[n_d]}}\right|+R_{2,n_2}(\mathcal{A}),\cdots,\\ &\left|a_{1^{[n_1]}\cdots 1^{[n_d]}}\right|+R_{d,1}(\mathcal{A}),\left|a_{2^{[n_1]}\cdots 2^{[n_d]}}\right|+R_{d,2}(\mathcal{A}),\cdots,\left|a_{n_d^{[n_1]}\cdots n_d^{[n_d]}}\right|+R_{d,n_d}(\mathcal{A}).\end{aligned}$$

Depending on whether $R_{k,i}(\mathcal{A})$ is 0, we can divide the above numbers into two categories and sort them:

① $R_{k,i}(\mathcal{A})\neq 0$:

$$\begin{aligned}\left|a_{k_1^{[n_1]}\cdots k_1^{[n_d]}}\right|+R_{t_{k_1},k_1}(\mathcal{A})\geq&\left|a_{k_2^{[n_1]}\cdots k_2^{[n_d]}}\right|+R_{t_{k_2},k_2}(\mathcal{A})\geq\cdots\\ \geq&\left|a_{k_p^{[n_1]}\cdots k_p^{[n_d]}}\right|+R_{t_{k_p},l_q}(\mathcal{A}),\end{aligned}$$

② $R_{k,i}(\mathcal{A})=0$:

$$\left|a_{l_1^{[n_1]}\cdots l_1^{[n_d]}}\right| \geq \left|a_{l_2^{[n_1]}\cdots l_2^{[n_d]}}\right| \geq \cdots \geq \left|a_{l_q^{[n_1]}\cdots l_q^{[n_d]}}\right|,$$

where $p+q=n_1+n_2+\cdots+n_d$, $k_1,k_2,\cdots,k_p$ are not necessarily different from each other, so are $l_1,l_2,\cdots,l_q$ and $t_{k_1},t_{k_2},\cdots,t_{k_p}$. For $i\in[q]$, $j\in\left[\max_{j\in[d]}\{n_j\}\right]$, $l_i\neq j$,

$$\left\{z\in\mathbb{R}:\left(|z|-\left|a_{l_i^{[n_1]}\cdots l_i^{[n_d]}}\right|\right)\left(|z|-\left|a_{j^{[n_1]}\cdots j^{[n_d]}}\right|\right)\leq 0\right\}\subseteq F(\mathcal{A})\subseteq G(\mathcal{A}).$$

For $i,j\in[p], i<j$, by Lemma 2.2,

$$\begin{aligned}&\left\{z\in\mathbb{R}:\left(|z|-\left|a_{k_i^{[n_1]}\cdots k_i^{[n_d]}}\right|\right)\left(|z|-\left|a_{k_j^{[n_1]}\cdots k_j^{[n_d]}}\right|\right)\leq R_{t_{k_i},k_i}(\mathcal{A})R_{t_{k_j},k_j}(\mathcal{A})\right\}\\ \subseteq&\left\{z\in\mathbb{R}:|z|-\left|a_{k_i^{[n_1]}\cdots k_i^{[n_d]}}\right|\leq R_{t_{k_i},k_i}(\mathcal{A})\right\}.\end{aligned}$$

Hence, $B(\mathcal{A})\subseteq G(\mathcal{A})$. □

Finally, we give an example.

Example 4.1. Let $\mathcal{A}\in\mathbb{C}^{10\times 8\times 5\times 7}$ be a complex tensor, where $(\mathcal{A})_{8726}=\frac{1}{\sqrt{6}},(\mathcal{A})_{9543}=\frac{1}{\sqrt{3}},(\mathcal{A})_{1221}=\frac{1}{\sqrt{6}}i,(\mathcal{A})_{3812}=-\frac{1}{\sqrt{3}}$, and other elements are 0. The maximum U-eigenvalue of $\mathcal{A}$ is 0.5774 [13]. After calculation, we obtain the following U-eigenvalue inclusion sets:

the Geršgorin-type inclusion set is

$$G(\mathcal{A})=\left\{\lambda\in\mathbb{C}:|\lambda|\leq\frac{\sqrt{6}}{3}\approx 0.8165\right\},$$

the Brauer-type inclusion set is

$$B(\mathcal{A})=\left\{\lambda\in\mathbb{C}:|\lambda|\leq\sqrt{\frac{\sqrt{2}}{3}}\approx 0.6866\right\}.$$

Acknowledgements

This work was supported in part by the National Natural Science Foundation of China (No. 12071097), the Joint Guidance Natural Science Foundation of Heilongjiang Province of China (No. LH2021A004) and the Basic Scientific Research Foundation of National Defense (No. JCKYS2021604SSJS002).

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