# Empirical Study on Option Pricing under Markov Regime Switching Economics

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**Abstract.** In this research, we summarize the results of a practical study of index options based on the option valuation model which was proposed by Siu and Yang (Acta Math. Appl. Sin. Engl. Ser., 25(3) (2009), pp. 339–388), where an EMM kernel is integrated which takes into account all risk components of a regime-switching model. Further, the regime-switching risk of an economy in the options is priced using a hidden Markov regime-switching model with the risky underlying asset being modulated by a discrete-time, finite-state, hidden Markov chain whose states represent the hidden states of an economy. We apply such a model to the pricing of Hang Seng Index options based on the real-world financial data from October 2009 to October 2010 (i.e., for the year in which the model was proposed). We employed the entropy martingale measure (EMM) approach proposed by Siu and Yang (Acta Math. Appl. Sin. Engl. Ser., 25(3) (2009), pp. 339–388) to determine the optimal martingale measure for the Markov-modulated GBM. In addition, we have proposed a numerical technique called the weighted difference method to compliment the EMM approach. We have also verified the extended jump-diffusion model under regime-switching that we proposed recently (Int. J. Finan. Eng., 6(4) (2019), 1950038) using the 50ETF options which are obtained from Shanghai Stock Exchange covering a time span from January 2018 to December 2022. Further, we have highlighted the challenges for the EMM kernel-based Markov regime-switching model for pricing the out-of-the-money index options in the real world.

**AMS subject classifications**: 91G20, 91G60

**Key words**: Option pricing, EMM, regime-switching, hidden Markov model, Esscher transform, weighted difference method.

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# 1   Introduction

In recent years, option valuation problems under regime-switching have received considerable interest in literature. A key feature of regime-switching models is that model parameters are modulated by a Markov chain whose states represent states of business cycles (see Hamilton (1989)). Some early works on option pricing under regime-switching conditions include Naik (1993), Guo (2001), Buffington and Elliott (2002), Elliott et al. (2005), Siu (2008) and others. To be more specific, Guo (2001) investigated an option pricing problem in an incomplete market modelled by adjoining the Geometric Brownian Motion (GBM) for stock returns with a Markov chain in a Black-Scholes (1973) economy. Buffington and Elliott (2002) considered the option pricing problems for European and American options in a Black-Scholes market in which the states of the economy are described by a continuous-time, finite-state, Markov chain. Yao et al. (2003) investigated the pricing of European options under a Markov-modulated GBM and determined an equivalent martingale pricing measure for the Markov-modulated GBM. Elliott et al. (2005) proposed the use of a regime-switching version of the Esscher transform to determine an equivalent martingale measure for valuing options in a Markov-modulated Black-Scholes-Merton economy. Indeed, Gerber and Shiu (1994) pioneered the use of the Esscher transform in finance, in particular in option valuation. It provides a convenient method to specify an equivalent martingale measure. Siu (2008) justified the use of the Esscher transform for option valuation in a regime-switching diffusion model and a regime-switching jump-diffusion model using a game theoretic approach. Siu and Yang (2009) considered a modified version of the Esscher transform used in Elliott et al. (2005) to incorporate explicitly the intensity matrix of the Markov chain in the specification of an equivalent martingale measure. Siu (2011) demonstrated, through a rigorous mathematical proof, that an optimal equivalent martingale measure selected by minimizing the relative entropy between an equivalent martingale measure and the real-world probability measure does not price the regime-switching risk. Elliott et al. (2013) investigated the pricing of both European and American-style options when the price dynamics of the underlying risky assets are governed by a Markov-modulated constant elasticity of variance process. Liu (2017) conducted an empirical study using Markov-modulated regime switching model on Hang Seng index options when the regime switching risk is priced. In recent years, regime-switching models have been extended to include a jump-diffusion process (Momeya, e al., 2016), or price different types of options, for instance, bond options (Shen, et al., 2013), currency options (Bo, et al., 2010; Liu, 2019), and foreign equity options (Lian, et al., 2016; Fan, et al., 2014).

In terms of option valuation principles, it has been established (see, Harrison

and Kreps (1979) and Harrison and Pliska (1981) that the absence of arbitrage is equivalent to the existence of an equivalent martingale measure under which all discounted price processes are martingales. However, when the market is incomplete, there are more than one equivalent martingale measure. How to choose a consistent pricing measure from the set of equivalent martingale measures becomes an important problem. For this purpose, the minimal relative entropy approach is often employed to select an equivalent martingale measure from its canonical space. As discussed in Siu (2011), Miyahara (1996) was the first to introduce the minimal entropy martingale measure (MEMM) approach to select an equivalent martingale measure in an incomplete market. Nowadays, the MEMM approach has become one of the major approaches for option valuation in an incomplete market. The basic idea of the MEMM approach is to select an equivalent martingale measure so as to minimize the "distance" between an equivalent martingale measure and a real world probability measure described by their relative entropy. Consequently, the MEMM is the equivalent martingale measure which is closest to the real-world probability measure. For details about the MEMM approach for option valuation, interested readers may refer to works by Miyahara (2001), and Fujiwara and Miyahara (2003).

In this paper, we conduct the empirical studies on the pricing of both the Hang Seng Index Options (HSI) and the Shanghai 50ETF options by using the Esscher transform and the MEMM approach. We model the price dynamics of the underlying risky asset which are governed either by a Markov-modulated geometric Brownian motion using a novel model that was proposed by Siu et al. (2009) or a jump-diffusion process (Liu, 2019), in which the regime-switching risk was supposed to be priced. We assume that the drift and the volatility of the underlying asset are modulated by an observable continuous-time, finite-state Markov chain, whose states represent observable states of an economy. Unlike most of the previous works of model development, we pay more attention to the option pricing performance of the model.

The rest of the paper is organized as follows: The next section describes the model dynamics. In Section 2, we present the two-stage pricing method including the extended jump-diffusion process. In Section 3, we present the numerical examples for the computation of the option prices. We also present and discuss the results of numerical experiments. The final section concludes the paper.

## 2   The price dynamics

The main goal in this section is to illustrate the price dynamics which is dominated by a Markov-modulated geometric Brownian motion. Such a framework has been well documented in Elliott (1993), Elliott et al. (1994), and Siu and Yang (2009).

Consider the money account $B$ and stock $S$ in a financial model, we shall describe the price dynamics of these two assets. Firstly, we define the hidden Markov chain $\{X_t\}_{t\in T}$ on the complete probability space $(\Omega,\mathcal{F},\mathcal{P})$ with a finite $\mathcal{X}:=(x_1,x_2,\cdots,x_N)$, where $\mathcal{T}$ denotes the finite time horizon $[0,T]$ and $P$ denotes a real world probability measure. According to Elliott et al. (1994), the state space of $\{X_t\}_{t\in T}$ is defined by a finite set of unit vectors $\varepsilon:=\{e_1,e_2,\cdots,e_N\}$, where $e_i=(0,\cdots,1,\cdots,0)\in\mathcal{R}^N$. Then, Elliott (1993) and Elliott et al. (1994) provide the following semi-martingale decomposition for $\{X_t\}_{t\in T}$:

$$X_t=X_0+\int_0^t QX_Sds+M_t, \tag{2.1}$$

where $Q$ denotes rate matrix $[q_{ij}(t)]_{i,j=1,2,\cdots,N}$ and $\{M_t\}$ is an $\mathcal{R}^N$-valued martingale with respect to the filtration which generated by $\{X_t\}_{t\in T}$ and the measure $\mathcal{P}$.

Assume that $\{r_t\}_{t\in T}$ denotes the market interest rate of the money market account at time $t$. We suppose that

$$r_t:=r(t,X_T)=\langle r,X_t\rangle, \tag{2.2}$$

where $r:=(r_1,r_2,\cdots,r_N)\in\mathcal{R}^N$ with $r_i>0$ for each $i=1,2,\cdots,N$. Therefore, the price dynamic of money market account $\{B_t\}_{t\in T}$ is modeled by

$$B_t=e^{-rt}=\exp\left(-\int_0^t r_u du\right). \tag{2.3}$$

In addition, assume that $\{\mu_t\}_{t\in T}$ and $\{\sigma_t\}_{t\in T}$ are the appreciation rate and the volatility of stock $S$, respectively, which are defined as follows:

$$\mu_t:=\mu(t,X_T)=\langle\mu,X_t\rangle, \tag{2.4a}$$

$$\sigma_t:=\sigma(t,X_T)=\langle\sigma,X_t\rangle, \tag{2.4b}$$

where $\mu:=(\mu_1,\mu_2,\cdots,\mu_N)\in\mathcal{R}^N$ and $\sigma:=(\sigma_1,\sigma_2,\cdots,\sigma_N)\in\mathcal{R}^N$ with $\sigma_i>0$ for each $i=1,2,\cdots,N$. Then, we use the Markov-modulated geometric Brownian motion with jump to define the dynamic of underlying stock $\{s_t\}_{t\in T}$:

$$dS_t=\mu_t S_t dt+\sigma_t S_t dW_t, \quad S_0=s, \tag{2.5}$$

where $\{W_t\}_{t\in T}$ denote the standard Brownian motion on $(\Omega,\mathcal{F},\mathcal{P})$. Then the price dynamic of $S$ can be written as

$$S_t=S_u\exp(Y_t-Y_u), \tag{2.6}$$

where $Y_t$ denotes the logarithmic return of $S$ over the interval $[0,t]$, and

$$Y_t=\int_0^t\left(\mu_s-\frac{1}{2}\sigma_s^2\right)ds+\int_0^t\sigma_s dW_S. \tag{2.7}$$

# 3　The pricing method

In this section, we shall introduce the Esscher transform first. We will adopt the method of Esscher Transform to determine a martingale condition, for which the risk-neutral Esscher parameters will be estimated. Second, we will determine an optimal set of risk-neutral Esscher parameters for pricing options. Since, in general, there are more than one set of the parameters which can satisfy the martingale condition after the Esscher transform, therefore, we shall adopt the method of minimizing the maximum entropy between the real-world probability and an equivalent martingale measure.

## 3.1　The Esscher transformation

Esscher transformation (Esscher, 1932) is introduced to determine the martingale condition in the paper. It is a time-honored tool in the actuarial science, and the definition of the transform is that it takes a probability density $f(x)$ and transform it to the new probability density $f(x,h)$ with the parameter $h$. That is, for the probability density function $f(x)$, let $h$ be a real number such that

$$M(h) = \int_{-\infty}^{\infty} e^{hx} f(x) dx, \tag{3.1}$$

exists as the function in $x$, then

$$f(x,h) = \frac{e^{h \cdot x} f(x)}{M(h)} \tag{3.2}$$

is a probability density and called Esscher transform.

## 3.2　Option pricing under regime switching

In this section, the regime-switching Esscher Transform and risk-neutral Esscher parameters will be described. Let $\mathcal{G}_T$ be the $\sigma$-algebra $\mathcal{F}_t^X \bigvee \mathcal{F}_t^S$, which is generated by $\{X_t\}_{t \in T}$ and $\{s_t\}_{t \in T}$ under the $\mathcal{P}$-argumentation of natural filtrations. Moreover, let $\theta_t$ be the regime switching Esscher parameter at time $t$, which can be written as follows:

$$\theta_t := \theta(t, X(t)) = \langle \theta, X_t \rangle, \tag{3.3}$$

where $\theta := (\theta_1, \theta_2, \cdots, \theta_N) \in \mathcal{R}^N$. Following Elliott (1982), write

$$(\theta \cdot Y)_t = \int_0^t \theta(u) dY(u) \quad \text{for each} \quad t \in T.$$

Then we define the regime-switching Esscher Transform on $Q_\theta \sim \mathcal{P}$ on $\mathcal{G}_T$ as follows:

$$\frac{dQ}{d\mathcal{P}} := \frac{e^{(\theta \cdot Y)T}}{E\left[e^{(\theta \cdot Y)T}\big|_{X(0)}\right]} = \Lambda_T, \tag{3.4}$$

where $E[\cdot]$ denotes an expectation under $\mathcal{P}$. We then consider the European option with payoff $V(S_T)$ at maturity $T$. Therefore, the conditional price of option given $\mathcal{G}_{(t)}$ is :

$$V_t := E^\theta \left[ \exp\left( -\int_t^T r_u du \right) V(S_T) \Big|_{G_t} \right]. \tag{3.5}$$

When the $S_t = s$ and $X_t = x$, the value of option is:

$$V(t,s,x) = E^\theta \left[ \exp\left( -\int_t^T r_u du \right) V(S_T) \Big|_{S_t=s,\ X_t=x} \right]. \tag{3.6}$$

For a European call option, it can be evaluated as follow according to (3.6), i.e.,

$$C(0,S_0,X_0) = E^\theta \left[ \exp\left( -\int_0^t r_u du \right) (S_T - K)^+ \Big|_{S_0,\ X_0} \right]. \tag{3.7}$$

The function can be re-written as follow by using regime-switching Esscher Transform as proposed in Siu et al. (2009):

$$\begin{aligned}
C(0,S_0,X_0) &= E\left[ \frac{dQ}{d\mathcal{P}} \exp\left( -\int_0^t r_u du \right) (S_T - K)^+ \Big|_{S_0,\ X_0} \right] \\
&= \frac{E\left[ e^{(\theta \cdot Y)T} \exp\left( -\int_0^T r_u du \right) (s_T - K)^+ \Big|_{s_0,X_0} \right]}{E[e^{(\theta \cdot Y)T}|_{X_0}]}.
\end{aligned} \tag{3.8}$$

We will use Monte Carlo simulations to estimate the call option prices. First, we divided the time horizon $[0,T]$ into $N$ subintervals $[t_j, t_{j+1}]$, $(j = 0, 1, \cdots, J-1)$ with equal length $\Delta t = T/J$, where $t_0 = 0$ and $t_J = T$. Then, for each $l = 1, 2, \cdots, L$, simulate the discrete-time version of Markov chain $X$ and obtain $\{X_{t_j}^{(l)}\}_{j=1}^J$ and its' corresponding $\{\mu_{t_j}^{(l)}\}_{j=1}^J$, $\{\gamma_{t_j}^{(l)}\}_{j=1}^J$, $\{\theta_{t_j}^{(l)}\}_{j=1}^J$, and $\{\sigma_{t_j}^{(l)}\}_{j=1}^J$, Finally, the $Y_{t_{(j+1)}}$ is defined as $f$:

$$Y_{t_{j+1}} = Y_{t_j} + \left( \mu_{t_j} - \frac{1}{2}\sigma_{t_j}^2 \right)\Delta + \sigma_{t_j}\xi_{t_{j+1}}, \tag{3.9}$$

where $\left\{\xi_{t_{j+1}}\right\}_{j=0,1,\cdots,j-1}$ and $\xi_{t_{j+1}} \sim N(0,\Delta)$. The parameters in Eq. (3.8) can be obtained in practice except for the risk-neutral Esscher parameters $\theta_t$. Therefore, in the next section we will present the method to calculate $\theta_t$.

## 3.3 Determination of risk-neutral Esscher parameters

First, we need to define a $(\mathcal{G},\mathcal{P})$-martingale $\{\Lambda_t\}_{t\in\mathcal{T}}$ as defined in Siu and Yang, (2009), i.e.,

$$\Lambda_t := E\left[\Lambda_T \middle| \mathcal{G}_t\right], \quad t\in\mathcal{T}. \tag{3.10}$$

Let

$$\tilde{S} := e^{-\int_0^t r(u)du} S(t)$$

for each $t\in T$. Here, the martin is given by considering an enlarged filtration as follows:

$$\widetilde{S}(u) = E^\theta[\widetilde{S}(t)|\mathcal{G}(u)] \quad \text{for any} \quad t,u\in T \quad \text{with} \quad t\geq u,$$

where $E^\theta$ denotes expectation under $Q_\theta$.

**Lemma 3.1.** *Define*

$$\lambda_i(\theta_i) := \theta_i\mu_i - \frac{1}{2}\theta_i\sigma_i^2 + \frac{1}{2}\theta_i^2\sigma_i^2, \quad i=1,2,\cdots,N, \tag{3.11a}$$

$$\widetilde{\lambda}_t(\theta_i) := -r_i + (\theta_i+1)\mu_i - \frac{1}{2}(\theta_i+1)\sigma_i^2 + \frac{1}{2}(\theta_i+1)^2\sigma_i^2, \tag{3.11b}$$

*where $\lambda(\theta) := (\lambda_1(\theta_1),\lambda_2(\theta_2),\cdots,\lambda_N(\theta_N))\in\mathcal{R}^N$ and $\overline{\lambda}(\theta) := (\overline{\lambda}_1(\theta_1),\overline{\lambda}_2(\theta_2),\cdots,\overline{\lambda}_N(\theta_N))$, $i=1,2,\cdots,N$. Then, the martingale condition is satisfied if and only if*

$$\left\langle e^{(Q+\mathrm{diag}(\tilde{\lambda}(\theta))(t-u)}X_u, 1_N\right\rangle - \left\langle e^{(Q+\mathrm{diag}(\lambda(\theta))(t-u)}X_u, 1_N\right\rangle = 0 \tag{3.12}$$

*for all $X_u$ and for all $t,u\in T$ with $t\geq u$.*

The proof of the above theorem employs a version of Bayes' rule and the definition of $\Lambda_t$ in Eq. (3.10), and can be found in Siu et al. (2009) and Elliott and Osakwe (2006), so we don't repeat here.

To expanse the term $\left\langle e^{(Q+\mathrm{diag}(\tilde{\lambda}(\theta))(t-u)}X_u, 1_N\right\rangle$, we will use the equation

$$\exp(M) = \sum_{n=0}^\infty \frac{M^n}{n!}.$$

The first-order approximation may be used to estimate the risk-neutral Esscher parameters $(\theta_1,\theta_2,\cdots,\theta_N)$ that corresponds to the Esscher parameters generated in the works by Elliott et al. (2005), whilst the second-order approximation may be used to estimate $(\theta_1,\theta_2,\cdots,\theta_N)$ as well. Siu and Yang (2009) proved that the

Esscher parameters can first be evaluated by Eq. (3.13) when using the first-order approximation of $\exp(M)$, i.e.,

$$\theta_i = \frac{r_i - \mu_i}{\sigma_i^2} \quad \text{for} \quad i^{\text{th}} \text{ economic state.} \tag{3.13}$$

In this paper, apart from using the first order approximation, we will also use the second-order approximation to estimate the risk-neutral Esscher parameters. Consequently, there will be more than one pair of $(\theta_1, \theta_2)$ in the latter case when Eq. (3.12) is solved for the regime-switching problem of two states. The min-max entropy method will therefore be used to select an optimal pair of $(\theta_1, \theta_2)$.

## 3.4 Relative entropy for equivalent martingale measure

The concept of entropy plays an important role in mathematical finance. Miyahara (1999) was the first to introduce the minimal entropy martingale measure (MEMM) approach to select an equivalent martingale measure in an incomplete market. Nowadays, the MEMM approach has become one of the major approaches for option valuation in an incomplete market. As we have discussed before, there are more than one set of $(\theta_1, \theta_2, \cdots, \theta_N)$ satisfying Eq. (3.12). We will choose an optimum set of risk-neutral Esscher parameters $(\theta_1, \theta_2, \cdots, \theta_N)$ by minimizing the maximum entropy between an equivalent martingale measure and the real world probability measure over different states. The principle of maximum entropy indicate that the probability distribution which best represents the current state of knowledge is the one with largest entropy. To maximize entropy, we should define the entropy between $Q_\theta$ and $\mathcal{P}$ conditional on $X_0 \in \varepsilon$. The entropy is defined as below

$$\begin{aligned} I\left(Q_\theta, \mathcal{P} \Big| X_0\right) &:= E\left[\frac{d\ell_\theta}{\mathcal{P}} \ln\left(\frac{d\ell_\theta}{\mathcal{P}}\right)\Big|_{X_0}\right] \\ &= \frac{E\left[(\theta \cdot Y)_T e^{(\theta \cdot Y)} T\Big|_{X_0}\right]}{E\left[e^{(\theta \cdot Y)T}\Big|_{X_0}\right]} - \ln E\left[e^{(\theta \cdot Y)T}\Big|_{X_0}\right] \\ &= \frac{\left\langle e^{(Q + \text{diag}(\lambda_z(\theta))T} X_0, 1_2\right\rangle}{\left\langle e^{(Q + \text{diag}(\lambda(\theta))T} X_0, 1_2\right\rangle} - \ln\left\langle e^{(Q + \text{diag}(\lambda(\theta))T} X_0, 1_2\right\rangle, \end{aligned} \tag{3.14}$$

where $X_0 = e_i$, $i = 1, 2$ for state 1 and state 2.

Define the $I(Q_\theta | \mathcal{P})$ is the maximum entropy between $Q_\theta$ and $\mathcal{P}$ over the different values $X_0$,

$$I\left(Q_\theta \Big| \mathcal{P}\right) := \max_{i=1,2,\cdots,N} I\left(Q_\theta, \mathcal{P} \Big|_{X_0 = e_i}\right). \tag{3.15}$$

Note that $N = 2$ in our research.

Then, a set of risk-neutral Esscher parameters are selected when $I(Q_\theta | \mathcal{P})$ is minimized.

## 3.5   Regime-switching jump-diffusion model

We have recently extended Siu and Yang's model to its Markov-modulated jump-diffusion model (see, Liu, 2019) in order to price the regime-switching risk for currency options, which is given by:

$$\frac{dS_t}{s_{t-}} = (\alpha_t - k\lambda_{t\Gamma})dt + \sigma_t dW_t + (e^{z_{t-}} - 1)dN_t, \tag{3.16}$$

Where $\{W_t\}_{t \in T}$ denote the standard Brownian motion on $(\Omega, \mathcal{F}, \mathcal{P})$, and the appreciation rate $\{\alpha_t\}_{t \in T}$, the stochastic volatility $\{\sigma_t\}_{t \in T}$ of the underlying and the stochastic jump intensity $\{\lambda_t\}_{t \in T}$ of the Poisson process $N = \{N_t\}_{t \in T}$ are all modulated by a common continuous-time, finite-state Markov chain $\xi = \{\xi_t\}t \in T$.

We decompose the log return of the underlying asset $Y_t = \log(S_t/S_0)$, $(0 \leq t \leq T)$ into a continuous part and a jump part

$$Y_t = C_t + J_t, \tag{3.17}$$

where $C_t$ and $J_t$ are the continuous diffusive part and the jump part of $Y_t$, and they admit the following forms:

$$C_t = \int_0^t \left( \alpha_s - k\lambda_s - \frac{1}{2}\sigma_s^2 \right) ds + \int_0^t \sigma_s dW_S, \tag{3.18a}$$

$$J_t = \int_0^t Z_{s-} dN_S. \tag{3.18b}$$

Then, we can write two family Esscher parameters $\{\theta_t^c\}_{t \in T}$ and $\{\theta_t^J\}_{t \in T}$ as follows:

$$\theta_t^c := \langle \theta^c, \xi_t \rangle, \tag{3.19a}$$

$$\theta_t^J := \langle \theta^J, \xi_t \rangle, \tag{3.19b}$$

where $\theta^c := (\theta_1^c, \theta_2^c, \cdots, \theta_N^c) \in \mathcal{R}^N$, and $\theta^J := (\theta_1^J, \theta_2^J, \cdots, \theta_N^J) \in \mathcal{R}^N$.

Note that we now have an enlarged set of Esscher parameters, i.e., $(\theta^c, \theta^J)$ satisfying the martingale condition in general. We need to choose an optimum set of risk-neutral Esscher parameters by minimizing the maximum entropy between an equivalent martingale measure and the real-world probability measure over different states. Such a process is similar to that summarized in Sections 2.2-2.4.

## 3.6   An alternative method for selecting an equivalent martingale measure

As shown in the previous sections, the basic idea of the MEMM approach is to select a proper equivalent martingale measure so as to minimize the "distance" between an equivalent martingale measure and the real-world probability measure described by their relative entropy. Consequently, the MEMM is the equivalent martingale measure which is closest to the real-world probability measure. However, in practice, the entropy may not be evaluated as the term $E[e^{(\theta \cdot Y)T} | X_0]$ is estimated by

$$
\left\langle e^{(Q+\text{diag}(\lambda(\theta))T} X_0, 1_2 \right\rangle
\approx \left\langle \left( I + Q + \text{diag}(\lambda_i(\theta_i)) T + \left( Q + \text{diag}(\lambda_i(\theta_i))^2 T^2 \right) X_0, 1_2 \right\rangle,
$$

which can become negative for some Esscher parameters $\theta_t$. To overcome this difficulty, we propose an alternative empirical method called the "weighted difference method" to directly minimize the "distance" between an equivalent martingale measure and the real world probability measure. Denote $B_t = r_t - \frac{1}{2}\sigma_t^2$, and let $\theta_t$ be the regime switching Esscher parameters at time $t$, which solves the martingale condition. Let $Y_t$ be the logarithmic return of $S$ as defined in (2.7), and the $I(Q_\theta | \mathcal{P})$ be the relative entropy between $Q_\theta$ and $\mathcal{P}$ as defined in (3.14), then we minimize the following parameter $d$ by a set of Esscher parameters $\theta_t^*$, that is

$$
d\big|_{\theta = \theta_t^*} := \min E^\theta \left[ \int_0^t \left( \theta_s (Y_s - B_s) - Y_s \right) ds \right]. \tag{3.20}
$$

The proof can be found in the appendix. In the equation, the term $\theta_s(Y_s - B_s)$ represents the "weighted" returns of the risky underlying asset at time $s$ under an equivalent martingale measure described by $\theta_s$, while $Y_s$ is the observed returns of the asset under the real world probability measure at time $s$. The expected value of the modulus of the differences between these two quantities conditional on $X_0$ over the time interval $[0, T]$ can represent the "explicit distance" between an equivalent martingale measure and a real world probability measure. The optimal equivalent martingale measure is the one that corresponds to the selected Esscher parameters which minimizes the "distance" $d$.

# 4   Numerical experiments

**Example 4.1.** Let us consider some specimen values for option pricing under regime switching. In our case, we suppose that there are two states in economy ($N = 2$).

State 1 denotes a "high state" economy while State 2 denotes a "low state" economy. The transition probability is $(I+Q\Delta)$, where

$$Q = \left[ \begin{array}{cc} q_{11} & q_{12} \\ q_{21} & q_{22} \end{array} \right], \quad q_{11} = -q_{12} = -\eta \quad \text{and} \quad q_{21} = -q_{22} = \eta.$$

The model parameters are given by:

$$\begin{aligned} r_1 &= 0.05, & (\mu_1, \sigma_1) &= (0.35, 0.1), \\ r_1 &= 0.01, & (\mu_2, \sigma_2) &= (0.05, 0.2), \\ \eta &= 0.5. \end{aligned}$$

It is straight forward to obtain the Esscher parameters for the case that the martingale condition is approximated by the first order expansion of $\exp(M)$, which are $\theta_1 = -30$, and $\theta_2 = -1$.

Now consider that the second order approximation of $\exp(M)$, and let $T = 0.5$ year and $\Delta = 0.00025$, we can obtain the following two equations using Eq. (3.12)

$$0.0000125\theta_1^3 + 0.0013\theta_1^2 + 0.031\theta_1 + 0.145 + 0.0025\theta_2 = 0,$$
$$0.0002\theta_2^3 + 0.0007\theta_2^2 + 0.0182\theta_2 + 0.03645 + 0.000625\theta_1 = 0.$$

Solve the equations

| $\left(\hat{\theta}_1, \hat{\theta}_2\right)$ | $I\left(Q_\theta \mid \mathcal{P}\right)$ | Weighted differen |
|---|---|---|
| $(-30, -1)$ | 0.2572 | 1285.2 |
| $(-5.8083, -1.8658)$ | 0.0498 | 372.5 |
| $(-64.1894, 0.1999)$ | 0.5471 | 3059.0 |

The above results are a little different from those obtained by Siu and Yang (2009) for the same example, however, we believe that the roots of the equations being solved should include a pair of $(\hat{\theta}_1, \hat{\theta}_2)$ that is for the case when the first order approximation is used (see, Elliott et al. (2005)). According to the principle of minimizing the maximum entropies, we select a set of risk-neutral Esscher parameters which can give minimum $I(Q_\theta \mid \mathcal{P})$. Hence, we pick the set of solution $(\hat{\theta}_1, \hat{\theta}_2) = (-5.80826, -1.86580)$. Similarly, we can choose the same set of Esscher parameters that correspond to the minimum $d$ that is given by the weighted difference method. In Table 1, some other cases have studied and the results are summarized there. It has clearly demonstrated that the weighted difference method can produce the same results as the minimizing the maximum entropy approach. This means that the weighted difference method can help to pick the right Esscher parameters that describe the optimal equivalent martingale measure.

Table 1: Esscher parameter selection by the MEMM and the weight difference method.

| options | Esscher parameters $(\hat{\theta}_1, hat\theta_2)$ | Entropy $(I_1, I_2)$ | Max Entropy | Weighted difference $d$ | Selected $(\hat{\theta}_1, hat\theta_2)$ |
|---|---|---|---|---|---|
| $K=80$, $T=0.25$ | (-56.174, -0.564) | (0.4804, 0.0042) | 0.4804 | 366.062 | (-13.826, -1.270) |
| | (-30.0, -1.0) | (0.2567, 0.0004) | 0.2567 | 328.453 | |
| | (-13.826, -1.270) | (0.1187, 0.0011) | $0.1187^\sharp$ | $119.852^*$ | |
| $K=100$, $T=0.75$ | (-66.45, 0.949) | (0.5684, 0.0003) | 0.5684 | 6716.5 | (-3.541, -2.499) |
| | (-30.0, -1.0) | (0.2576, 0.0022) | 0.2576 | 2378.5 | |
| | (-3.541, -2.499) | (0.0301, 0.0012) | $0.0301^\sharp$ | $489.9^*$ | |
| $K=110$, $T=1.0$ | (-30.0, -1.0) | (0.2572, 0.0014) | 0.2572 | 5121.0 | (-2.435, -3.128) |
| | (-67.540, 1.603) | (0.5794, 0.0012) | 0.5794 | 1256.0 | |
| | (-2.435, -3.128) | (0.0208, 0.0017) | $0.0208^\sharp$ | $628.0^*$ | |

$*$   The minimum $d$ based on which the Esscher parameters are selected using the weighted difference method.

$\sharp$   The minimum max-entropy based on which the Esscher parameters are selected using MEMM.

Table 2: Call option prices with first-order and second order approximations ($S_0=100$). FI–First order approximation; and SE–Second order approximation.

| Strike price, $K$ | $T=0.25$ | | $T=0.5$ | | $T=0.75$ | | $T=1.0$ | |
|---|---|---|---|---|---|---|---|---|
| | FI | SE | FI | SE | FI | SE | FI | SE |
| 80 | 21.63 | 21.86 | 21.78 | 25.94 | 22.51 | 32.04 | 23.07 | 35.48 |
| 90 | 11.16 | 15.08 | 11.95 | 17.59 | 12.57 | 23.59 | 13.33 | 25.35 |
| 100 | 2.24 | 3.23 | 2.12 | 7.51 | 2.59 | 12.14 | 3.54 | 18.05 |
| 110 | 0 | 0 | 0.07 | 2.13 | 0.04 | 6.18 | 0.05 | 10.79 |
| 120 | 0 | 0 | 0 | 0 | 0.003 | 2.63 | 0.0028 | 6.25 |
| 130 | 0 | 0 | 0 | 0 | 0 | 0.0003 | 0.0005 | 2.31 |

Finally, options with different strike prices and maturity times in this example are priced and the results are given in Table 2.

As seen in the table, the option prices evaluated are usually higher when the second order approximation is used than those using the first approximation. This is consistent with the findings in the work of Siu et al. (2009). The usual effects of strike price and maturity time on option price apply here, i.e., The larger the strike price is, the lower the option price is, and the longer the maturity time is, the higher the option price is. The results have demonstrated that both the MEMM approach and the weighted difference method are effective for deciding the correct Esscher parameters, and, the subsequent option prices.

**Example 4.2** (HK HSI Options). In the part, we present a real data example to illustrate the problem with the MEMM approach and the application of the weighted difference method for determination of an optimal equivalent martingale measure. We then compare the call prices estimated by the model with the market prices for

different strike prices and maturities.

We use a data set of daily closing prices of Hang Seng Index (HSI), from 31 October 2009–31 October 2010, which was retrieved from the HK stock exchange for the year when the model was just established. There are in total 252 observations. In this investigation, the number of regime states is taken to be two. The estimated Markov regime-switching parameters are

$$(\mu_1,\sigma_1)=(0.0017,0.0084), \qquad r_1=0.007,$$
$$(\mu_2,\sigma_2)=(-0.0003,0.0131), \qquad r_2=0.007.$$

The transition probabilities are estimated to be

$$P=\begin{pmatrix} 0.99 & 0.01 \\ 0.05 & 0.95 \end{pmatrix}.$$

We extend the work of Siu et al. (2009) so that the model can deal with the cases when the rate matrix are controlled by two different components, i.e., the rate matrix components can be calculated as follows,

$$q_{12}=-q_{11}=-\frac{P_{12}\ln(1-P_{12}-P_{21})}{\Delta(P_{12}+P_{21})},$$
$$q_{21}=-q_{22}=-\frac{P_{21}\ln(1-P_{12}-P_{21})}{\Delta(P_{12}+P_{21})}.$$

Suppose the current time is $t_0$. Without loss of generality, we put $t_0=0$ and $S_0$ (the index HIS) is 23,652.94 as observed on 1 November 2010 on the HK stock exchange.

Firstly, we present the results of selecting optimal martingale measures in Table 3 for some typical cases using both MEMM and the Weighted difference approaches. It shows that the MEMM approach fails to identify a suitable martingale measure for option valuation for all the cases shown in Table 3. Taking the case that $K=21,000$ and $T=0.417$ as an example, there are three pairs of Esscher parameters $(\hat{\theta}_1,\hat{\theta}_2)$ that satisfy the equivalent martingale conditions Eq. (3.12), i.e., $(75.11,42.54)$, $(-1889.67,1507.73)$, and $(1968.81,-1500.49)$. However, the computed entropy (i.e., $I_1$) corresponding to one of the Esscher parameters (i.e., $\hat{\theta}_1$) given by Eq. (3.14) is a complex number. This causes difficulty in deciding a set of maximum entropies based on which the min-max entropy needs to be selected in order to identify an optimal martingale measure. In contrast, the weighted difference method gives three different "distance" values, i.e., 15.59, 418.79, and 428.79, from which the minimum "distance" is chosen to be 15.59. Then, the right Esscher parameters $(\hat{\theta}_1,\hat{\theta}_2)$ are selected as $(75.11,42.54)$. The option prices are then evaluated using the selected Esscher parameters.

Table 3: Esscher parameter selection by the MEMM and weight difference method.

| Options | Esscher parameters $(\hat{\theta}_1, hat\theta_2)$ | Entropy $(I_1, I_2)$ | Max Entropy | Weighted difference, $d$ | Selected $(\hat{\theta}_1, hat\theta_2)$ |
|---|---|---|---|---|---|
| $K=21000$, $T=0.417$ | (75.11, 42.54) | (-7.15- 3.14i, -7.16) | N/A | 15.59 | (75.11, 42.54) |
| | (-1889.67, 1507.73) | (-3.56, -1.84) | -1.84 | 418.79 | |
| | (1968.81, -1500.49) | (-4.04, -2.18) | -2.18 | 428.17 | |
| $K=21400$, $T=0.67$ | (75.11, 42.54) | (-8.11-3.14i, -8.11) | N/A | 41.5 | (75.11, 42.54) |
| | (1974.33,-1504.03) | (-4.97, -3.09) | -3.09 | 1133.3 | |
| | (-1895.64,1511.26) | (-4.48, -2.75) | -2.75 | 1098.6 | |
| $K=22000$, $T=0.917$ | (75.11, 42.54) | (-8.74-3.14i, -8.74) | N/A | 77.0 | (75.11, 42.54) |
| | (-1898.30,1512.82) | (-5.10, -3.36) | -3.36 | 2062.7 | |
| | (1976.78,-1505.59) | (-5.58, -3.70) | -3.70 | 2114.7 | |

Table 4: Market and estimated HSI option values at $t_0=0$ for different strike prices and maturities.

| Strike | Market price ($T=0.25$) | $T=0.25$ | $T=0.417$ | $T=0.67$ | $T=0.917$ |
|---|---|---|---|---|---|
| 20,800 | 3,102 | 3,229.4 | 3,231.3 | 3,232.8 | 3,235.5 |
| 21,200 | 2,757 | 2,830.3 | 2,832.2 | 2,835.0 | 2,838.2 |
| 21,600 | 2,424 | 2,430.8 | 2,433.4 | 2,437.3 | 2,440.9 |
| 22,000 | 2,104 | 2,031.7 | 2,034.2 | 2,038.7 | 2,043.3 |
| 22,400 | 1,799 | 1,632.3 | 1,636.0 | 1,641.4 | 1,646.2 |
| 22,800 | 1,505 | 1,232.8 | 1,236.7 | 1,242.7 | 1,248.3 |
| 23,200 | 1,250 | 834.1 | 838.4 | 844.5 | 851.1 |
| 23,600 | 973 | 434.6 | 439.0 | 446.6 | 453.2 |
| 24,000 | 821 | 35.0 | 40.2 | 47.9 | 55.8 |
| 24,400 | 653 | 0.0 | 0.0 | 0.0 | 0.0 |

Finally, a number of Hang Seng index options for 1 November 2010 have been evaluated and the results are summarized in Table 4 in together with their market prices. The strike prices range from 20,800 to 24,400, and the maturities of the options are 3 months, 5 months, 8 months, and 11 months respectively. It can be seen that, firstly, the regime-switching model yields comparable results with the market option prices, especially, for the options in the money. General trends of the option prices along with the strike prices and the maturity times seem reasonable, for instance, the index option's price decreases when the strike price increases in all the cases studied, whilst it increases slightly when the maturity time becomes longer.

**Example 4.3** (Shanghai 50ETF options). The data of 50ETF options are acquired from Shanghai Stock Exchange covering a time span from January 2018 to December 2022. The option type is European option. The following table (Table 5) shows some typical option contracts with their trading particulars.

There are in total 75,873 call option contracts in the dataset. Suppose that the hidden Markov chain has two states, which means that we consider the domestic macroeconomic shifts between two states: $e_1$ ("good") and $e_2$ ("bad") in economy.

Table 5: Sample 50ETF options.

| code | open | high | low | close | $K$ | $t$ |
|------|------|------|------|-------|-----|-----|
| 510050C1803M02650 | 0.4831 | 0.4991 | 0.4507 | 0.455 | 2.65 | 0.230 |
| 510050C1803M03100 | 0.1042 | 0.1131 | 0.0803 | 0.0845 | 3.1 | 0.230 |
| 510050C1803M03200 | 0.0594 | 0.066 | 0.044 | 0.0469 | 3.2 | 0.230 |
| 510050C1803M03300 | 0.0329 | 0.037 | 0.023 | 0.025 | 3.3 | 0.230 |
| 510050C1803M03400 | 0.0184 | 0.0203 | 0.0121 | 0.0136 | 3.4 | 0.230 |
| 510050C1803M03500 | 0.0106 | 0.0116 | 0.0068 | 0.0075 | 3.5 | 0.230 |
| 510050C1806M03400 | 0.0675 | 0.0725 | 0.0562 | 0.0567 | 3.4 | 0.59 |

Table 6: The estimated parameter values (for similarity, we assume jumps follow the same distribution in both states).

| Parameter name | Value in state $e_1$ | Value in state $e_2$ |
|----------------|----------------------|----------------------|
| Annual interest rate | $r_1=0.025$ | $r_2=0.015$ |
| Volatility | $\sigma_1=0.1$ | $\sigma_2=0.3$ |
| Appreciation of $S$ | $\mu_1=0.61$ | $\mu_2=-0.18$ |
| Annual jump intensity | $\lambda_1=7$ | $\lambda_2=9$ |
| Mean jump size | $\mu_J=0.06$ | $\mu_J=0.06$ |
| Standard deviation of the jump size | $\mu_J=0.10$ | $\mu_J=0.10$ |

We estimate that the transition probability matrix of the two state Markov chain is given by

$$\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} 0.99 & 0.01 \\ 0.03 & 0.97 \end{pmatrix}.$$

Moreover, we shall adopt the values for the model parameters, which are given in Table 6. The parameter values related to the continuous diffusive component and the jump component.

Denote the current time is $t_0$, and $S_0$ is 2.818 as observed on 1 Auguat 2022 on the 50ETF (510050). For each of the fixed maturity years $T=0.0575$, 0.06301, 0.1589, and 0.40548 years, we are concerned with a range of strike price $K$ from 2.65 to 3.5 RMB where appropriate. Taking the case of $T=0.0575$ and $K=2.65$ for example, it is straight forward to obtain the Esscher parameters for the case that regime-switching risk is not priced, which are $\theta_1^c=-11.4886829$, $\theta_2^c=8.882569$, $\theta_1^J=-6.50$, and $\theta_2^J=-6.50$. For the case of pricing regime-switching risk under the jump-diffusion process, we can compute the maximum entropies and the Esscher

parameters as follows:

| $\left(\theta_1^c,\theta_2^c,\theta_1^J,\theta_2^J\right)$ | $I\left(Q_\theta\middle|\mathcal{P}\right)$ |
|---|---|
| (48.176979,-8.82816,-6.5,-6.5) | 0.768779 |
| (-11.48868,8.882569,-6.5,-6.5) | 1.3778387 |
| (-73.05528, -26.31811, -6.5, -6.5) | 2.941906 |

According to the principle of minimizing the maximum entropies, we select a set of risk-neutral Esscher parameters which can give a minimum $I(Q_\theta|\mathcal{P})$. Hence, we pick the Esscher parameters $\left(\theta_1^c,\theta_2^c,\theta_1^J,\theta_2^J\right)=(48.176979,-8.82816,-6.5,-6.5)$ for determining the equivalent martingale measure. The above process will yield two different call option prices, i.e., the call price without regime-switching risk is 0.13814265, and the call price with regime-switching risk is 0.1323195, whilst the market close price is quoted as 0.13.

Finally, we follow the similar approach to price more ETF50 options, and their computed prices are given in Table 7 together with their market prices. The results in Table 7 confirm that the jump-diffusion model under Markov regime-switching is effective in pricing Chinese options.

Table 7: Market and estimated ETF50 option values at $t_0=0$ for different strike prices and maturities.

| Strike | $T=0.0575$ | | $T=0.06301$ | | $T=0.1589$ | | $T=0.40548$ | |
|---|---|---|---|---|---|---|---|---|
| | Market | Estimat | Market | Estimat | Market | Estimat | Market | Estimat |
| 2.65 | 0.130 | 0.133997 | 0.1398 | 0.13538 | 0.1661 | 0.1552 | 0.2192 | 0.199692 |
| 2.7 | 0.0907 | 0.090219 | 0.1014 | 0.090322 | 0.1348 | 0.124369 | 0.1891 | 0.167401 |
| 2.75 | 0.0614 | 0.059964 | 0.0687 | 0.060187 | 0.1025 | 0.09658 | 0.1625 | 0.144212 |
| 2.8 | 0.0371 | 0.03527 | 0.0433 | 0.038049 | 0.0771 | 0.075261 | 0.1379 | 0.131922 |
| 2.85 | 0.0211 | 0.02528 | 0.0254 | 0.026161 | 0.0566 | 0.059245 | 0.1157 | 0.114996 |
| 2.9 | 0.0123 | 0.020513 | 0.0147 | 0.021045 | 0.0401 | 0.046739 | 0.096 | 0.095054 |
| 2.95 | 0.0063 | 0.016197 | 0.0077 | 0.017 | 0.0282 | 0.040693 | 0.08 | 0.083803 |
| 3.0 | 0.0038 | 0.013342 | 0.0034 | 0.012909 | 0.0137 | 0.034684 | 0.0535 | 0.07437 |

However, in the above two examples, the regime-switching models (either with Jump-diffusion or not) do not give good performance for the out-of-the-money options. This is true with both the Hong Kong Hangsheng index option data and Shanghai ETF50 option data. As seen in Fig. 1, the prediction error can be as large as 279% for some ETF50 options. The finding is consistent with what was found in the works by Liew et al. (2010) that their regime-switching option model could not give good results for certain options such as out-of-the-money options.

The explanations on such deviations can be vary. One may contribute the cause to estimations of regime-switching parameters, real-world arbitrage opportunities
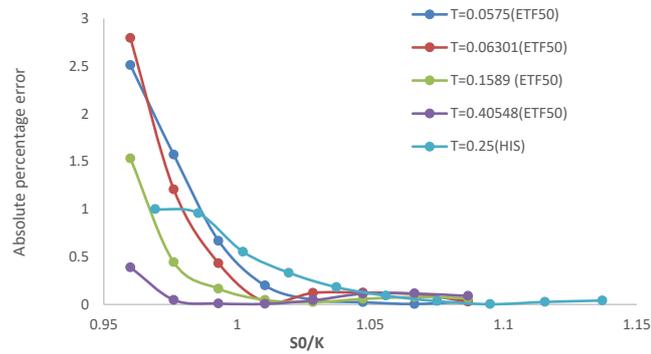
Figure 1: The absolute percentage errors between estimated option prices and market prices as a function of Moneyness Ratio $(S_0/X)$.

(see, for example, Liew et al. (2010)), financial market characteristics, and most importantly, model bias. In reality, there may have existed a significant market sentiment that the market index could go higher in the near future. Although our study on the index options using the Markov regime-switching model is preliminary, and is restricted to the financial data studied, we assume that a highlight of such an issue may be beneficial for the better development of pricing models in terms of their accurate option pricing in the real world. Indeed, there are recent studies (Zghal, et al. 2020; Li, et al. 2021) that are in consideration of market sentiment and information asymmetry so that there will be reasonable corrections to the model mispricing of options in real market.

# 5   Conclusions

In conclusion, the main purpose of the paper is to conduct an empirical analysis of index options in real-worlds, namely Hang Seng Index (HIS) options and ETF50 options, using the framework of the Markov regime switching model that was originally proposed by Siu et al. (2009) and extended by Liu (2019), where the price dynamics of the risky underlying asset is modulated by a hidden Markov chain of finite number of states. In the study, we have addressed the potential problem with the method of maximum entropy martingale measure (MEMM) adopted for determining the Esscher parameters for an optimal equivalent martingale measure. We find that such an approach may not work well in some real-world option pricing applications due to the difficulty in obtaining a meaningful maximum entropy $I\big(Q_\theta\big|,\mathcal{P}|X_0\big)$. We have further proposed an alternative empirical remedy, i.e., the weighted difference method, for the purpose of estimating the "distance" directly between an equiva-

lent martingale measure and the real-world probability measure. We show that this approach is efficient in selecting the right Esscher parameters in the cases that the MEMM may not be capable of. Some theoretical proof for the proposed method needs to be developed. Further, through the empirical applications of both Siu and Yang's model and the Markov jump-diffusion model to index options, we have also highlighted the current challenges for the Markov Regime-Switching models in the real-world problems for pricing the out-of-the-money options.

# Appendix

**Proposition A.1.** *Denote $B_t = r_t - 1/2\sigma_t^2$, and let $\theta_t$ be the regime switching Esscher parameters at time $t$, which solves the martingale condition. Let $Y_t$ be the logarithmic return of $S$ as defined in (2.7), and the $I(Q_\theta|\mathcal{P})$ be the relative entropy between $Q_\theta$ and $\mathcal{P}$ as defined in (3.14), then the lower bound of the entropy $I(Q_\theta|\mathcal{P})$ is the following minimized parameter $d$ by a set of Esscher parameters $\theta_t^*$, that is*

$$d\big|_{\theta=\theta_t^*} := \min E^\theta \left[ \int_0^t (\theta_s(Y_s - B_s) - Y_s) ds \right]. \tag{A.1}$$

*Proof.* By the definition of $\frac{dQ}{d\mathcal{P}}$, we have the following

$$
\begin{aligned}
I(Q_\theta|\mathcal{P}) := & E^\theta \left[ \ln\left( \frac{dQ}{d\mathcal{P}} \right) \right] = E^\theta \left[ \ln\left\{ \frac{e^{(\theta \cdot Y)_t}}{E\left[e^{(\theta \cdot Y)_t} \mid \mathcal{F}_t^\xi\right]} \right\} \right] \\
\geq & E^\theta \left[ \ln\left\{ \exp\left[ \int_0^t \theta_s \left( \mu_s - \frac{1}{2}\sigma_s^2 \right) ds + \int_0^t \theta_s \sigma_s dW_S - \int_0^t \theta_s \left( \mu_s - \frac{1}{2}\sigma_s^2 \right) ds \right. \right.\right. \\
& \left.\left.\left. - \int_0^t (\theta_s \sigma_s)^2 ds \right] - \exp\left( \int_0^t Y_s ds \right) \right\} \right] \\
= & E^\theta \left[ \int_0^t \theta_s Y_s ds - \int_0^t Y_s ds - \int_0^t \theta_s \left( \mu_s - \frac{1}{2}\sigma_s^2 \right) ds - \int_0^t (\theta_s \sigma_s)^2 ds \right] \\
= & E^\theta \left[ \int_0^t \theta_s Y_s ds - \int_0^t Y_s ds - \int_0^t \theta_s \left( \mu_s - \frac{1}{2}\sigma_s^2 + \theta_s \sigma_s^2 \right) ds \right] \\
= & E^\theta \left[ \int_0^t \theta_s Y_s ds - \int_0^t Y_s ds - \int_0^t \left[ \theta_s \left( \mu_s + \theta_s \sigma_s^2 - r_s \right) + \theta_s \left( r_s - \frac{1}{2}\sigma_s^2 \right) \right] ds \right].
\end{aligned}
$$

Due to the martingale condition (3.12), we have

$$
\begin{aligned}
E^\theta\left[\ln\left(\frac{dQ}{d\mathcal{P}}\right)\right] &\geq E^\theta\left[\int_0^t \theta_s Y_s ds - \int_0^t Y_s ds - \int_0^t \theta_s\left(r_s - \frac{1}{2}\sigma_s^2\right)ds\right] \\
&= E^\theta\left[\int_0^t \theta_s\left[Y_s - \left(r_s - \frac{1}{2}\sigma_s^2\right)\right]ds - \int_0^t Y_s ds\right] \\
&= E^\theta\left[\int_0^t \left\{\theta_s\left[Y_s - \left(r_s - \frac{1}{2}\sigma_s^2\right)\right] - Y_s\right\}ds\right].
\end{aligned}
$$

Since the left-hand side is the relative entropy, it is clearly seen that the minimization of $E^\theta[\int_0^t(\theta_s(Y_s - B_s) - Y_s)ds]$ yields the lower bound of the relative entropy $I(Q_\theta|\mathcal{P})$. This proves the proposition. $\qquad\square$

**Remark A.1.** First, in Eq. (A.1), the term $\theta_t Y_t$ represents the "weighted" returns of the risky underlying asset at time t under an equivalent martingale measure described by $\theta_t$, while $Y_t$ is the "observed" returns of the asset under the real world probability measure at time $t$. The expected value of the modulus of the difference terms over a time interval $[0,t]$ can represent the "explicit distance" between an equivalent martingale measure and the real world probability measure. The optimal equivalent martingale measure is the one that corresponds to the selected Esscher parameters which minimize the "distance" $d$. This gives a "better" sense of meaning than the concept of relative entropy. Second, $B_t = r_t - 1/2\sigma_t^2$. The quantity $\theta_t B_t$ represents the combined "contributions" by the risk-free rate and asset price volatility to the asset returns under an equivalent martingale measure described by $\theta_t$. In practice, if both the risk-free rate and the price volatility (or $B_t$ itself) are sufficiently small, Eq. (A.1) can be reduced to:

$$
d = E^\theta\left[\int_0^t (\theta_s Y_s - Y_s)ds\right].
$$

The method is useful when pricing an option under Markov-regime switching using Esscher transform as an equivalent martingale measure needs to described by $\theta_t$.

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