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A Modified Weak Galerkin Method for Stokes Equations

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Abstract. In this paper, we modify the weak Galerkin method introduced in [15] for Stokes equations. The modified method uses the $\mathbb{P}_k/\mathbb{P}_{k-1}$ $(k\geq 1)$ discontinuous finite element combination for velocity and pressure in the interior of elements. Especially, the numerical traces v_{hb} which are defined in the interface of the elements belong to the space $C^0(\mathcal{E}_h)$, this change leads to less degree of freedom for the resultant linear system. The stability, priori error estimates and L^2 error estimates for velocity are proved in this paper. In addition, we prove that the modified method also yields globally divergence-free velocity approximations and has uniform error estimates with respect to the Reynolds number. Finally, numerical results illustrate the performance of the method, support the theoretical properties of the estimator and show the efficiency of the algorithm.

AMS subject classifications: 65M60, 65N30

Key words: Weak Galerkin, Stokes equations, globally divergence-free, less degree of freedom, uniform error estimates.

1 Introduction

The Galerkin mixed method is an efficient numerical method for solving the Stokes equations. Unfortunately, the method requires the pair of finite element spaces for the velocity and pressure to satisfy an inf-sup stability condition (see, e.g., [1, 2, 8, 14]). Many stabilized finite element methods have been proposed to circumvent the inf-sup difficulty,

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e.g., Galerkin least-square methods [12, 13, 23, 27], pressure projection methods [9, 17, 24] and so on.

Moreover, the numerical solution of the finite element algorithms is required to satisfy the incompressible condition (at least locally). However, it is not easy for the continuous finite element space. More recently, the discontinuous Galerkin (DG) method has been developed and analyzed for the Stokes equations and Navier-Stokes equations. This method have some good features like local conservation of physical quantities and flexibility in meshing. Furthermore, a class of divergence-free discontinuous Galerkin (DG) methods were proposed in [4, 18] by using $H(div)$-conforming elements. In addition, the local discontinuous Galerkin (LDG) methods [3] and hybridizable discontinuous Galerkin (HDG) methods [5–7, 22] are developed for the Stokes and Navier-Stokes equations. However, all of these methods use a postprocessing procedure [5,7, 22] or use element-wise divergence-free spaces for velocity [6] to enforce the incompressibility.

The weak Galerkin (WG) finite element method was fist proposed by Wang and Ye for second-order elliptic problems [19, 20]. It has the flexibility of using discontinuous finite element functions, and the differential operators ∇ are approximated by weak forms ∇_w which is locally-defined on each element. Then, this method was applied to the Stokes equations [21], Darcy-Stokes equations [25], Convection-Diffusion-Reaction equations [16], N-S equations [28], Soblev equation [11], etc. In [15], a globally divergence-free WG method which has uniform error estimates with respect to the Reynolds number for Stokes equations was proposed and analyzed. According to the above discussion, in this paper, motive by [15], we are interested in modifying this WG method for Stokes problem. We also use the $\mathbb{P}_k/\mathbb{P}_{k-1}$ $(k\geq 1)$ discontinuous finite element combination for approximating of the velocity and pressure, and the piecewise $\mathbb{P}_k$ for numerical traces of velocity and pressure. The difference is in the modified method the numerical traces $\boldsymbol{v}_{hb}$ which are defined in the interface of the elements belong to the space $C^0(\mathcal{E}_h)$. The analysis of the original method can not completely inherit to the modified method. This change makes the local projection not satisfy the commutation properties which are important in the analysis of the stability and error estimates. In this paper, we will prove that the modified method is stable and also yields globally divergence-free velocity approximations. We derive a series of uniform error estimates which are independent of the Reynolds number. The numerical examples are presented to illustrate the effectiveness and the theoretical properties of algorithm. Most importantly, the numerical results show the modified WG method has less degree of freedoms.

The rest of this paper is organized as follows. In Section 2, we propose a modified WG finite element scheme for the Stokes equations. In Section 3, we give the definition and properties of local projections which is very important in the analysis of the stability and error estimates. In Section 4, we prove the stability of the modified WG method. In Section 5, we derive a series of error estimates. In Section 6, we show some results of the numerical experiments.