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An Immersed Finite Element Method for Elliptic Interface Problems with Multi-Domain and Triple Junction Points

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Abstract. Interface problems have wide applications in modern scientific research. Obtaining accurate numerical solutions of multi-domain problems involving triple junction conditions remains a significant challenge. In this paper, we develop an efficient finite element method based on non-body-fitting meshes for solving multi-domain elliptic interface problems. We follow the idea of immersed finite element by modifying local basis functions to accommodate interface conditions. We enrich the local finite element space by adding new basis functions for handling non-homogeneous flux jump. The numerical scheme is symmetric and positive definite. Numerical experiments are provided to demonstrate the features of our method.

AMS subject classifications: 35R05, 65N15, 65N30

Key words: Immersed finite element, interface problems, triple junction, multi-domain.

1 Introduction

The elliptic interface problems have raised much attention in the past decades and related numerical methods have been developed for solving interface problems accurately and efficiently. Conventional numerical methods, such as finite element method (FEM) [8], can be used to solve interface problems. These methods require the mesh to fit the interface; thus they are sometimes called fitted-mesh methods. A limitation of these methods

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is that the solution mesh has to be regenerated when dealing with a moving interface problem because of this body-fitting restriction.

Many numerical methods based on non-body-fitting meshes have been developed for solving interface problems. In the finite difference framework, since the pioneering work of Peskin [25] on the immersed boundary method, there have been the immersed interface method [20], the matched interface and boundary (MIB) method [30], the kernel-free boundary integral method [29], the embedded boundary method [16] and the cut-cell method [19]. In the finite element framework, there are generalized FEM [4], extended FEM [9], cut FEM [12] and the immersed finite element (IFE) method [21]. The IFE method was first introduced in [21] for one-dimensional interface problems with piecewise linear polynomial approximation. Since then, the method has been extended to multi-dimensional problems [10, 13, 22, 23], higher order approximations [2, 3, 6, 7] and interface problems of other partial differential equation models [1, 15, 24]. Moreover, the IFE method has also been extended to non-homogeneous flux jump conditions [11, 14].

So far, most IFE methods in the literature are designed to solve interface problems with two sub-domains. When it comes to multi-domain problems with intersecting interfaces, such as problems with triple interface junction points, the complexity of the problems will inevitably increase. There are a few numerical methods developed for multi-domain interface problems. In [17], the authors developed a Petrov-Galerkin type method for multi-domain problems with triple junction conditions and triple junction points, which is an extension from two-domain interface problems [18]. Another method is based on MIB framework [28] for scalar jumping coefficients. The bandwidth of the sparse linear system in [28] is usually larger than that in [17]. Some more recent work on multi-domain interface problems include the three dimensional problem in [26] and the elasticity problem in [27].

In this paper, we propose a numerical method based on IFE methods for solving the elliptic interface problem with triple-junction interfaces. We present the construction of IFE basis functions on interface elements with triple-junction points. Local and global IFE spaces will be formulated accordingly. Moreover, we extend the construction for handling the non-homogeneous flux jump following the idea in [14]. Compared with previous methods in [17], there are two improvements in the method in this paper. First, the construction of IFE basis function covers more types of geometrical configuration of interface elements. To be more specific, there is no need for the assumption in [17] that a triple junctional interface must intersect with three different edges of an element. The second advantage of this method is that the stiffness matrix of linear system is always symmetric positive definite, compared with the non-symmetric positive definite matrix in [17].

The rest of article is organized as follows. In Section 2, we introduce the elliptic interface problem with multi-domain and triple junction interfaces and derive its weak formulation. In Section 3, we construct the local IFE basis functions and the non-homogeneous flux basis functions on different types of interface elements. Then we formulate the local and global IFE spaces and apply them for solving the multi-interface problem. In Section