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On Diagonal Form Fast Multipole Method for an Oscillatory Boundary Integral Equation

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Abstract. We compare the diagonal form fast multipole method (FMM) with the traditional boundary element method (BEM) for a boundary integral equation (BIE) with oscillatory Hankel kernels which arising in using hybrid numerical-asymptotic boundary integral method to the two-dimensional (2D) scattering of a time-harmonic acoustic incident plane wave. The diagonal form FMM is a very efficient and popular algorithm for the rapid solution of boundary value problems. However, we show that the efficiency of the diagonal form FMM is greatly reduced for this kind of BIE. Numerical examples are given to confirm the proposed results.

AMS subject classifications: 65D32, 65D30

Key words: Oscillatory Hankel kernels, highly oscillatory integrals, Clenshaw-Curtis Filon methods, oscillatory integral equations.

1 Introduction

Various problems of mathematical physics and engineering can be described by partial differential equations and these problems can often be reformulated as an equivalent integral equation over the boundary [1,2]. Especially, in the computation of 2D acoustic scattering problems, the scattering of time-harmonic acoustic waves can be formulated as the Helmholtz equation subject to appropriate boundary conditions [3–6],

$$\Delta u(x)+\omega^2 u(x)=0, \quad x\in R^2\setminus\bar{\Omega}, \tag{1.1}$$

where ω is the wave number defined by $\omega=\hat{f}/c$, in which $\hat{f}$ is the angular frequency.

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The solution to Eq. (1.1) can be given as follows

$$u(x)=\int_{\Gamma}u(y)\frac{\partial G(x,y)}{\partial n(y)}-G(x,y)\frac{\partial u}{\partial n}(y)ds(y)+u^i(x),\quad x\in R^2\backslash\bar{\Omega}, \tag{1.2}$$

with

$$G(x,y)=\frac{i}{4}H_0^{(1)}(\omega|x-y|), \tag{1.3}$$

where $H_0^{(1)}$ denotes the Hankel function of the first kind and $n(y)$ is the outward normal vector at point y, Γ is the boundary of obstacle Ω, u^i is an incident wave.

Letting point x approach the boundary leads to the following traditional BIE

$$c(x)u(x)=\int_{\Gamma}u(y)\frac{\partial G(x,y)}{\partial n(y)}-G(x,y)\frac{\partial u}{\partial n}(y)ds(y)+u^i(x),\quad x\in\Gamma, \tag{1.4}$$

where constant $c(x)=1/2$ if Γ is smooth around point x.

The existence and uniqueness of a solution $u\in C(\bar{\Omega})\cap C^2(\Omega)$ follow from classical results, see [4,7]. For a 2D exterior acoustic problem, the Sommerfeld radiation condition

$$\lim_{r\to\infty}r^{1/2}\Big(\frac{\partial u^s}{\partial r}-i\omega u^s\Big)=0,\quad r:=|x|,$$

on the scattered field $u^s:=u-u^i$ should be added to ensure that the solution of (1.1) is an outgoing wave.

In the last few years, many numerical methods have been developed to solve the BIE (1.4), among which BEM also referred to as the BIE method, has been widely used to solve acoustic problems for many years. The BEM only needs to discretize the boundary instead of the domain, which make the mesh generation faster. Unfortunately, the BEM leads to systems of equations with dense and non-symmetrical coefficient matrices with $\mathcal{O}(N^2)$ elements need to be stored, where N being the number of degrees of freedom. Solving the BEM systems of equations directly will need $\mathcal{O}(N^3)$ arithmetic operations. To overcome this drawback, Rokhlin and Greengard innovate FMM [2,8,9], which has been widely used for solving large-scale engineering problems such as potential, elastostatic, Stokes flow, and acoustic wave problems. However, the kernel in (1.3) is highly oscillatory when $\omega\gg 1$ and (weakly) singular. Then the standard BEM or FMM may suffer from difficulty for computation the solution of (1.4) since the computation of highly oscillatory integrals by standard quadrature methods is exceedingly difficult and the cost steeply increases with the frequency [10–12].

On the other hand, because of the rapid oscillation of solution (1.4) when ω is large, the number of degrees of freedom grows at least linearly with respect to the wavenumber ω for standard numerical schemes. Recently, the high-frequency asymptotics of the solution was incorporated into the approximation space to reduce the computation cost.