Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **12**, No. 1, pp. 1-29

DOI: 10.4208/aamm.OA-2019-0124
February 2020

High-Order Accurate Entropy Stable Finite Difference Schemes for One- and Two-Dimensional Special Relativistic Hydrodynamics

Junming Duan[1] and Huazhong Tang[1,2,*]

[1] *Center for Applied Physics and Technology, HEDPS, and LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, China*
[2] *School of Mathematics and Computational Science, Xiangtan University, Xiangtan, Hunan 411105, China*

Received 27 April 2019; Accepted (in revised version) 12 July 2019

Abstract. This paper develops the high-order accurate entropy stable finite difference schemes for one- and two-dimensional special relativistic hydrodynamic equations. The schemes are built on the entropy conservative flux and the weighted essentially non-oscillatory (WENO) technique as well as explicit Runge-Kutta time discretization. The key is to technically construct the affordable entropy conservative flux of the semi-discrete second-order accurate entropy conservative schemes satisfying the semi-discrete entropy equality for the found convex entropy pair. As soon as the entropy conservative flux is derived, the dissipation term can be added to give the semi-discrete entropy stable schemes satisfying the semi-discrete entropy inequality with the given convex entropy function. The WENO reconstruction for the scaled entropy variables and the high-order explicit Runge-Kutta time discretization are implemented to obtain the fully-discrete high-order entropy stable schemes. Several numerical tests are conducted to validate the accuracy and the ability to capture discontinuities of our entropy stable schemes.

AMS subject classifications: 65M10, 78A48

Key words: Entropy conservative scheme, entropy stable scheme, high order accuracy, finite difference scheme, special relativistic hydrodynamics.

---

1 Introduction

This paper is concerned with the high-order accurate numerical schemes for the one- and two-dimensional special relativistic hydrodynamic (RHD) equations, which in the

*Corresponding author.
Emails: duanjm@pku.edu.cn (J. M. Duan), hztang@math.pku.edu.cn (H. Z. Tang)

laboratory frame, can be cast in the divergence form

$$\frac{\partial \boldsymbol{U}}{\partial t}+\sum_{\ell=1}^{d}\frac{\partial \boldsymbol{F}_\ell(\boldsymbol{U})}{\partial x_\ell}=0, \tag{1.1}$$

where $\boldsymbol{U}$ and $\boldsymbol{F}_\ell$ are respectively the conservative vector and the flux vector in the x_ℓ-direction and defined by

$$\boldsymbol{U}=(D,m_1,\cdots,m_d,E)^{\mathrm{T}}, \tag{1.2a}$$

$$\boldsymbol{F}_\ell=(Du_\ell,m_1u_\ell+p\delta_{1,\ell},\cdots,m_du_\ell+p\delta_{d,\ell},m_\ell)^{\mathrm{T}}, \qquad \ell=1,\cdots,d, \tag{1.2b}$$

with the mass density $D=\rho W$, the momentum density $\boldsymbol{m}=(m_1,\cdots,m_d)^{\mathrm{T}}=DhW\boldsymbol{u}$, and the energy density $E=DhW-p$. Here, $d=1$ or 2, ρ, p and $\boldsymbol{u}=(u_1,\cdots,u_d)^{\mathrm{T}}$ denote the rest-mass density, the kinetic pressure, and the fluid velocity, respectively. Moreover, $W=1/\sqrt{1-(u_1^2+\cdots+u_d^2)}$ is the Lorentz factor and h is the specific enthalpy defined by $h=1+e+p/\rho$ with units in which the speed of light is equal to one, and the specific internal energy e. The system (1.1)-(1.2) should be closed by using the equation of state (EOS). This paper will only consider the ideal-fluid EOS

$$p=(\Gamma-1)\rho e$$

with the adiabatic index $\Gamma\in(1,2]$. Because there is no explicit expression for the primitive variables $(\rho,\boldsymbol{u}^{\mathrm{T}},p)$ and the flux $\boldsymbol{F}_\ell$ in terms of $\boldsymbol{U}$, in order to recover the values of the primitive variables and the flux from the given $\boldsymbol{U}$, a nonlinear algebraic equation such as

$$E+p=DW+\frac{\Gamma}{\Gamma-1}pW^2$$

has to be numerically solved to obtain the pressure p, and then the rest-mass density ρ, the specific enthalpy h, and the velocity $\boldsymbol{u}$ can be orderly calculated by

$$\rho=\frac{D}{W}, \qquad h=1+\frac{\Gamma p}{(\Gamma-1)\rho}, \qquad \boldsymbol{u}=\frac{\boldsymbol{m}}{Dh}.$$

The relativistic description for the dynamics of the fluid (gas) at nearly the speed of light should be considered when we investigate the astrophysical phenomena from stellar to galactic scales, e.g., coalescing neutron stars, core collapse supernovae, active galactic nuclei, superluminal jets, the formation of black holes, and gamma-ray bursts etc. Due to the relativistic effect, the nonlinearity of the system (1.1)-(1.2) becomes much stronger than the non-relativistic case so that its analytic treatment is extremely difficult and challenging. Numerical simulation is a primary way to help us understand the physical mechanisms in the RHD. It can be traced back to the artificial viscosity method for the RHD equations in the Lagrangian coordinates [29, 30] and in the Eulerian coordinates [40]. After those, the modern shock-captured methods for the RHD