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Perfectly Matched Layer Method for Acoustic Scattering Problem by a Locally Perturbed Line with Impedance Boundary Condition

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Abstract. In this paper, we study the two-dimensional Helmholtz scattering problem by a locally perturbed line with impedance boundary condition. Different from the problem with Dirichlet boundary condition, the Green function of the Helmholtz equation with impedance boundary condition becomes very complicated and comprises surface waves along the locally perturbed line. A uniaxial perfectly matched layer (UPML) method is proposed to truncate the half plane into a bounded computational domain. The main contribution of this paper is to prove the well-posedness of the PML problem and the exponential convergence of the approximate solution to the exact solution as either the thickness or the medium parameter of PML increases.

AMS subject classifications: 65N30, 78A45, 35Q60

Key words: Uniaxial perfectly matched layer, Helmholtz equation, locally perturbed half-plane, impedance condition.

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1 Introduction

Numerical solution of scattering problems has drawn considerable attentions for its broad real-life applications and mathematical interests. The treatment of radiation conditions of scattering solutions is the first key ingredient of numerical simulations. It involves the truncation of an unbounded domain to a bounded domain and imposes highly accurate boundary conditions at the artificial boundary (cf. e.g., [24–26, 34]). Recently, there arises a surge of studies on the scattering problems involving infinite boundaries, such as the scattering in layered media and half-spaces (cf. e.g., [11,19,20,22,35]). With the appearance of infinite boundaries, the scattering waves usually comprise reflective waves,

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transmitted waves and surface waves. Thus the numerical treatment of radiation conditions becomes very challenging and appeals for new theories and methods.

In this paper, we study the time-harmonic scattering problem governed by the Helmholtz equation with impedance boundary condition in a locally perturbed half-plane:

$$\Delta u+k^2u=0 \qquad \text{in } \mathbb{R}^2_{\Sigma+}, \tag{1.1a}$$

$$\frac{\partial u}{\partial \boldsymbol{n}}-\mathbf{i}k\beta u=g \qquad \text{on } \Sigma, \tag{1.1b}$$

$$\lim_{r=|\boldsymbol{x}|\to+\infty}\int_{S_r^1}\left|\frac{\partial u}{\partial r}-\mathbf{i}ku\right|^2=0, \qquad \lim_{r\to+\infty}\int_{S_r^2}\left|\frac{\partial u}{\partial r}-\mathbf{i}\sqrt{Z^2+k^2}\,u\right|^2=0, \tag{1.1c}$$

which could model outdoor sound propagation or the harbour resonances. For example, in the harbour resonances, the sea is supposed to fill the half-plane which is locally perturbed by harbour geometry. Here $k>0$ is the constant wave number, $\Sigma=\{(x_1,p(x_1)): x_1\in\mathbb{R}\}$ is the infinite boundary, $p\in C(\mathbb{R})$ is a piecewise C^1-smooth function supported in $[-1,1]$, $\mathbb{R}^2_{\Sigma+}:=\{\boldsymbol{x}\in\mathbb{R}^2: x_2>p(x_1)\}$ and $\boldsymbol{n}$ is the unit outer normal on Σ pointing to the exterior of $\mathbb{R}^2_{\Sigma+}$. For convenience, we write Σ into the combination of the flat part and the perturbed part

$$\Sigma=\Sigma_\infty\cup\Sigma_p, \qquad \Sigma_\infty:=\{(x_1,0):|x_1|\geq 1\}, \qquad \Sigma_p=\Sigma\backslash\Sigma_\infty.$$

Clearly Σ is a local perturbation of the horizontal axis $\Sigma_0:=\partial\mathbb{R}^2_+$, where $\mathbb{R}^2_+:=\{(x_1,x_2): x_1\in\mathbb{R},\ x_2>0\}$ (see Fig. 1 for an illustration of the setting). β is the acoustic admittance, which would have to be taken as a complex-valued piecewise constant function in order to model the boundaries of rocks, sand, concrete in the applications. Here $\beta\in L^\infty(\Sigma)$ and satisfies

$$\mathrm{Re}(\beta)\geq 0 \quad \text{and} \quad \beta\equiv-\mathbf{i}Z/k \quad \text{on } \Sigma_\infty,$$

where $Z>0$ is the constant impedance parameter. We assume that the boundary condition on Σ satisfies

$$g\in H^{-1/2}(\Sigma), \qquad \mathrm{supp}(g)\subset\Sigma_p.$$

The two radiation conditions in (1.1c) represent propagating waves and surface waves respectively (see [22]) where

$$S_r:=\{\boldsymbol{x}\in\mathbb{R}^2_+:|\boldsymbol{x}|=r\}, \qquad S_r^1:=\left\{\boldsymbol{x}\in S_r: x_2\geq r^{1/4}\right\}, \qquad S_r^2:=S_r\backslash S_r^1. \tag{1.2}$$

The existence and uniqueness of the solution to (1.1) are studied in [22]. Our concerns here are to propose an approximation of the radiation conditions on a truncation boundary and to solve the approximate problem on the bounded domain numerically.