Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **12**, No. 1, pp. 164-188

DOI: 10.4208/aamm.OA-2019-0088
February 2020

The Weak Galerkin Finite Element Method for Solving the Time-Dependent Integro-Differential Equations

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Received 1 April 2019; Accepted (in revised version) 8 October 2019

Abstract. In this paper, we solve linear parabolic integral differential equations using the weak Galerkin finite element method (WG) by adding a stabilizer. The semi-discrete and fully-discrete weak Galerkin finite element schemes are constructed. Optimal convergent orders of the solution of the WG in L^2 and H^1 norm are derived. Several computational results confirm the correctness and efficiency of the method.

AMS subject classifications: 65M60, 65M15, 65M12

Key words: Integro-differential problem, weak Galerkin finite element method, discrete weak gradient, discrete weak divergence.

1 Introduction

Integro-differential equations [24] are used to simulate many phenomena in the fields of mathematics, dynamics and engineering technology [1, 9, 15]. It is also used in high-energy physics and biomedicine to help describe related physical phenomena and laws [7, 13]. Especially, in geology, the integro-differential equation [31] can be used to describe the fine three-dimensional problem of the interior of the earth to explore mineral products and predict earthquakes [8]. It also plays an important role in aerodynamics [3]. For example, it blue can be used to study the Brown displacement and thermal diffusion of suspended grain in heterogeneous fluid. When determining the profile of airfoil, the integro-differential equation can be used to calculate the effect of circulation, lift and

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resistance of the air [10]. Because of their application values, integro-differential equations are essential and significant research subjects. In this paper, we consider the linear parabolic integro-differential equation in domain $\Omega\subset\mathcal{R}^2$ with boundary $\partial\Omega$: seek an unknown function $u=u(\mathbf{x},t)$ satisfying:

$$u_t(\mathbf{x},t)-\nabla\cdot(A\nabla u(\mathbf{x},t))-\int_0^t\nabla\cdot(B\nabla u(\mathbf{x},\zeta))d\zeta=f(\mathbf{x},t),\qquad(\mathbf{x},t)\in\Omega\times(0,T],\tag{1.1a}$$

$$u=g(\mathbf{x},t),\qquad(\mathbf{x},t)\in\partial\Omega\times(0,T],\tag{1.1b}$$

$$u(\mathbf{x},0)=\psi(\mathbf{x}),\qquad\mathbf{x}\in\Omega,\tag{1.1c}$$

where $\mathbf{x}=\{x_1,x_2\}$, $A=[a_{i,j}(\mathbf{x},t)]_{2\times2}$, $B=[b_{i,j}(\mathbf{x},t)]_{2\times2}$, $A_t=[(a_{i,j})_t(\mathbf{x},t)]_{2\times2}$ and $B_t=[(b_{i,j})_t(\mathbf{x},t)]_{2\times2}$. The matrix-valued functions A and B are sufficiently smooth and A is symmetric. They also satisfy the some properties with positive constants α_1, α_2, α_3, β_1, β_2 for any $\xi,\eta\in\mathcal{R}^2$ that

$$\alpha_1\|\xi\|^2\le\xi^TA\xi\le\alpha_2\|\xi\|^2,\qquad\|\xi^TA_t\eta\|\le\alpha_3\|\xi\|\|\eta\|,$$
$$\|\xi^TB\eta\|\le\beta_1\|\xi\|\|\eta\|,\qquad\|\xi^TB_t\eta\|\le\beta_2\|\xi\|\|\eta\|.$$

Several numerical methods for problem (1.1a)-(1.1c) have been proposed. The earliest ones are the finite element (FE) methods [2, 4, 25] and the finite volume (FV) methods [14, 16]. One important characteristic of the finite element method is that it can preserve the conservation of mass and momentum. This method is primarily applied for the diffusion problems and the existence and uniqueness are proved. However, the finite volume method is preferred to the finite element method for conservation and stability. And it is more suitable for the discretization of the conservation of laws. A FV-FE method [12] which combines the advantages of the above methods is proposed. However, these methods require two mutually associated meshes. In order to reduce this correlation, many experts and scholars have proposed various discontinuous Galerkin methods. However, it is difficult to construct the penalty items of the discontinuous Galerkin method [5, 11].

Wang and Ye in 2011 proposed the weak Galerkin finite element for the second-order elliptic equations [19, 29]. The method is applied to many problems, such as Stokes equations [17, 18, 20–22, 28], Brinkman problem [23, 27], Biharmonic equations [30], eigenvalue problems [26] and Stochastic problems [6, 33] and so on. The partition of the domain can be arbitrary polygonal or polyhedral. The construction of the approximated function is simple and satisfies the stability condition. The essence of the weak Galerkin finite element is that the classical operators are replaced by some weak operators. In paper [32], the weak Galerkin finite element method is applied to the linear parabolic integro-differential equations. It proposes the semi-discrete and fully-discrete weak Galerkin finite element schemes. The optimal error estimates are obtained.

In this paper, we propose another weak finite element method by adding a stabilizer. The reason why we propose this method is that: firstly, the approximation space is easily constructed and simply satisfies the stability condition; moreover, the element