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A Jacobi Collocation Method for the Fractional Ginzburg-Landau Differential Equation

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Abstract. In this paper, we design a collocation method to solve the fractional Ginzburg-Landau equation. A Jacobi collocation method is developed and implemented in two steps. First, we space-discretize the equation by the Jacobi-Gauss-Lobatto collocation (JGLC) method in one- and two-dimensional space. The equation is then converted to a system of ordinary differential equations (ODEs) with the time variable based on JGLC. The second step applies the Jacobi-Gauss-Radau collocation (JGRC) method for the time discretization. Finally, we give a theoretical proof of convergence of this Jacobi collocation method and some numerical results showing the proposed scheme is an effective and high-precision algorithm.

AMS subject classifications: 35R35, 65M12, 65M70

Key words: The fractional Ginzburg-Landau equation, Jacobi collocation method, convergence.

1 Introduction

The fractional Ginzburg-Landau equation (FGLE) is known as a generalization of the classical one and has been presented to depict many kinds of nonlinear phenomena. The Ginzburg-Landau equation (GLE) has a variety of applications, e.g., in biology and chemistry. In many areas of physics, the GLE also has important applications, such as superconductivity, superfluidity, nonlinear optics, Bose-Einstein condensation and so on [1].

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At first, Ginzburg and Landau proposed the GLE to depict phase transitions in superconductors near their critical temperature. The GLE also models the dynamics of electromagnetic behavior of a superconductor in an external magnetic field [2].

In recent years, as the fractional differential equations have many applications in different fields of engineering and science, it attracted more and more scholars. A fractional Ginzburg-Landau equation is derived by Tarasov et al. [2] from the variational Euler-Lagrange equation for fractal media. Because fractals generate in a fractal media or a fractal process in nature, the FGLE has been used to depict many physical phenomena, for example, the dynamical processes in continuum with fractal dispersion or in media with a fractal mass dimension [2], the organization of a system near the phase transition point influenced by a competing nonlocal ordering [4], and a network of diffusively Hindmarsh-Rose neurons with a long-range synaptic coupling [5].

In this paper, we consider a numerical algorithm for solving the following Ginzburg-Landau equation with fractional Laplace operator ($1<a\leq 2$):

$$u_t+(\nu+i\eta)(-\Delta)^{\frac{a}{2}}u+(k+i\zeta)|u|^2u-\gamma u=0, \qquad x\in\mathbb{R}, \qquad t\in[0,T], \tag{1.1}$$

and the initial condition

$$u(x,0)=u_0(x), \qquad x\in\mathbb{R}, \tag{1.2}$$

where $i^2=-1$, $u(x,t)$ is the unknown complex function from $\mathbb{R}\times\mathbb{R}^+$ to $\mathbb{C}$, $u_0(x)$ is a given smooth function, and $\nu>0$, $k>0$, η, ζ, γ are real constants. The fractional Laplace operator $(-\Delta)^{\frac{a}{2}}$ can be defined with the symbol $|\xi|^a$ as follows:

$$-(-\Delta)^{\frac{a}{2}}u(x,t)=-\mathcal{F}^{-1}(|\xi|^a\tilde{u}(\xi,t)), \tag{1.3}$$

where $\tilde{u}$ is the Fourier transform of u and $\mathcal{F}$ denotes the Fourier transform operator. Yang et al. [6,7] showed that the fractional derivative defined in Eq. (1.3) is equivalent to the Riesz fractional derivative, i.e.,

$$\begin{aligned}&\frac{\partial^a}{\partial|x|^a}u(x,t)=-(-\Delta)^{\frac{a}{2}}u(x,t)\\ =&-\frac{1}{2\cos(\frac{a\pi}{2})}\left[{}_{-\infty}^{\ R}D_x^a u(x,t)+{}_x^R D_{+\infty}^a u(x,t)\right].\end{aligned} \tag{1.4}$$

The left Riemann-Liouville fractional derivative of $u(x,t)$ is defined as follows:

$${}_{-\infty}^{\ R}D_x^a u(x,t)=\begin{cases}\dfrac{1}{\Gamma(2-a)}\dfrac{\partial^2}{\partial x^2}\displaystyle\int_{-\infty}^{x}(x-s)^{1-a}u(s,t)ds, & 1<a<2,\\ \dfrac{\partial^2}{\partial x^2}u(x,t), & a=2,\end{cases}$$