Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **12**, No. 2, pp. 480-502

DOI: 10.4208/aamm.OA-2019-0056
April 2020

Spectral Collocation Methods for Second-Order Volterra Integro-Differential Equations with Weakly Singular Kernels

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Received 28 February 2019; Accepted (in revised version) 23 September 2019

Abstract. In this paper, a Jacobi spectral collocation approximation is proposed for the solution of second-order Volterra integro-differential equations with weakly singular kernels. The solution of such equations usually exhibits a singular behaviour at the origin. By using some suitable variable transformations, we obtain a new equation which is still weakly singular, but whose solution is as smooth as we like. Then the resulting equation is solved by standard spectral methods. We establish a rigorous error analysis for the proposed method, which shows that the numerical errors decay exponentially in L^∞-norm and weighted L^2-norm. Finally, to perform the numerical simulation, a test example is considered with non-smooth solutions.

AMS subject classifications: 65R20, 65M70, 45D05

Key words: Volterra integro-differential equations, weakly singular kernel, spectral collocation methods.

1 Introduction

In this paper we shall be concerned with the approximate solution of the second-order

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weakly singular Volterra integro-differential equations (VIDEs)

$$y''(t)=a(t)y(t)+b(t)y'(t)+f(t)+\int_0^t (t-s)^{-\mu}K_1(t,s)y(s)ds$$
$$+\int_0^t (t-s)^{-\mu}K_2(t,s)y'(s)ds, \qquad t\in I:=[0,T], \tag{1.1a}$$
$$y(0)=y_0, \qquad y'(0)=y'_0, \tag{1.1b}$$

where, the functions $a,b,f:I\to\mathbb{R}$ and $K_1,K_2:D\to\mathbb{R}$ (with $D=:\{(t,s):0\le s\le t\le T\}$) denote given smooth functions, and constant μ satisfies $0<\mu<1$.

In the last decade, spectral methods for approximating the solution of Volterra integral or integro-differential equations have received considerable attention due to their high accuracy. We refer to [5, 20, 21] for a systematic introduction of spectral approximations and associated algorithms. It is known that spectral accuracy can be achieved with spectral collocation approximations in the case of linear and nonlinear Volterra integral or integro-differential equations with smooth kernels (see [1, 12, 25, 27, 32, 39]). This is also true for Volterra linear integral or integro-differential equations with weakly singular kernel $(t-s)^{\mu}$, $0<\mu<1$, under the assumption that the underlying solution is smooth enough (see [6,7,22,23,34,36,40]). In [8,41], a Jacobi spectral collocation method was proposed for linear Volterra integral equations with weakly singular kernels $(t-s)^{\mu}$, $0<\mu<1$, and with a nonsmooth solution. Some function transformations and variable transformations were employed to transform the equation into a new one which possessed better regularity, so that the orthogonal polynomial theory can be applied directly. In [24, 26], the Jacobi spectral collocation method was used to approximate the nonsmooth solution of nonlinear Volterra integral or integro-differential equations with weakly singular kernel $(t-s)^{\mu}$, $0<\mu<1$. And the authors in [14, 33] applied the Jacobi spectral collocation method to solve the particular case of linear Volterra integral or integro-differential equations, that is, with weakly singular kernels $(t-s)^{\frac{1}{2}}$, where a nonsmooth solution were considered. In the convergence analysis of these works, only variable transformations were used. In [11], the authors applied Chebyshev spectral collocation method to linear weakly singular Volterra integral equation with a nonsmooth solution directly, and took into account the singularity of the solutions in their convergence analysis. Spectral and pseudo-spectral Galerkin approaches were considered in [4,28,30,38] for smooth linear and nonlinear Volterra integral or integro-differential equations, and were extended in [15, 42] to linear weakly singular Volterra integral equations with a nonsmooth solution.

But the literature on the numerical solution of (1.1a)-(1.1b) is comparatively small. Moreover, as far as high-order VIDEs are concerned, very little convergence analysis has been given so far. Wei and Chen presented a study of spectral collocation techniques for Eqs. (1.1a)-(1.1b) with $\mu=0$ in [31], and for high-order linear VIDEs with smooth kernels in [35]. And we extended the spectral collocation method for high-order nonlinear VIDEs with smooth kernels in [25]. In the present paper, we develop a Jacobi spectral collocation