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A Posteriori Error Analysis of Any Order Finite Volume Methods for Elliptic Problems

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Abstract. In this paper, we construct and analyze the a posteriori error estimators for any order finite volume methods (FVMs) for solving the elliptic boundary value problems in R^2. We shall prove that the a posteriori error estimators yield the global upper and local lower bounds for the H^1-norm error of the corresponding FVMs. So that the a posteriori error estimators are equivalent to the true errors in a certain sense. Lots of numerical experiments are performed to illustrate the theoretical results.

AMS subject classifications: 65N30, 65N12

Key words: Any order finite volume methods, a posteriori error estimate.

1 Introduction

The FVM is widely used in scientific and engineering computations, especially in computational fluid dynamics (cf. [17, 19, 25, 26, 29]). Compared with other numerical methods for solving partial differential equations, preserving certain local conservation laws is the most attractive advantage of the FVM. Many researchers have studied this method extensively and obtained some important results. We refer to [2, 3, 5, 9, 15, 18, 20] for an incomplete list of references.

The a prior error analysis of high order FVMs has made significant progress in the past twenty years. In [9], a systematic framework was developed for analyzing the uniform ellipticity of the bilinear forms for FVMs which leads to the optimal error estimate of the methods in the H^1-norm. Paper [10] provided a method for the construction of any order FVMs based on triangular meshes so that the uniform ellipticity of the bilinear forms of the methods can be analyzed under a unified theoretical framework. The high order FVMs presented in [9, 10] enjoy the optimal error estimate in the H^1-norm under certain minimum angle conditions. Paper [34] used the second degree Gauss points to

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construct a special quadratic FVM and proved that the inf-sup condition of the scheme hold independently of the minimal angle condition of the underlying mesh. Paper [31] presented a unified analysis for the L^2-norm error estimate of the FVMs and constructed many high order schemes which enjoy the optimal L^2-norm error estimate. For FVMs over quadrilateral meshes, paper [33] constructed a family of any order FVMs and developed a unified proof for the inf-sup condition of the methods under the condition that the underlying mesh is an $h^{1+\gamma}$ parallelogram mesh ($\gamma>0$).

A posterior error estimate plays a crucial role in adaptive mesh procedures which have been very popular in numerical simulation in many fields. Developing the theory of a posterior error estimates and designing adaptive algorithms for the FVMs achieve much attention in recent years and important results are obtained for linear FV schemes, see [4,6,8,13,16,21–24,30]. To our best knowledge, there are no works so far about the a posterior error estimates for high order FVMs. The purpose of our paper is to construct and analyze residual type a posterior error estimators for any order FVMs over triangular meshes for solving the second elliptic boundary value problems. We shall prove the reliability and efficiency of the a posterior error estimators so that they are equivalent to the exact H^1-norm error of the FVMs in a certain sense. We know that the bilinear forms of the linear FVMs are the same as those of the linear finite element methods [14, 32]. However, this fact dose not hold for high order FVMs. In this paper, we shall choose two proper mappings: from the H^1 space to the trial space and from the trial space to the test space. The approximating properties of mappings and the properties of the cut-off functions play pivotal roles in the proof of the reliability and efficiency of the a posterior error estimators.

The rest of this paper is organized as follows. In Section 2, we present any order FVMs for solving the boundary problems of the second order partial differential equations. In Section 3, we define the a posteriori error estimators for any order FVMs and prove their reliability. In Section 4, we establish the efficiency of the a posterior error estimators. In the last section, we present numerical examples to confirm the theoretical results.

In this paper, "$A\lesssim B$" means that $A\leq CB$, where C is a constant independent of the parameters which A and B may depend on. We use $A\sim B$ to denote $A\lesssim B$ and $B\lesssim A$.

2 Any order FVMs for elliptic problems

We shall use the standard Sobolev notation. For a positive integer k and a subdomain $G\subseteq R^2$, let $H^k(G)$ denote the Sobolev space with the norm $\|\cdot\|_{k,G}$ and the corresponding semi-norm $|\cdot|_{k,G}$. Let $(\cdot,\cdot)_G$ and $||\cdot||_{0,G}$ denote the scalar product and the corresponding norm of $L^2(G)=H^0(G)$. When $G=\Omega$, we shall omit the index Ω for simplicity. By $H_0^1(G)$ we denote the subspace of $H^1(G)$ whose functions have the vanishing trace on ∂G.

We consider the Dirichlet problem of the second order partial differential equation

$$\begin{cases} -\nabla\cdot(\mathbf{a}\nabla u)=f & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$