

Constructing Separable Non- 2π -Periodic Solutions to the Navier-Lamé Equation in Cylindrical Coordinates Using the Buchwald Representation: Theory and Applications

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Abstract. In a previous paper (Adv. Appl. Math. Mech., 10 (2018), pp. 1025–1056), we used the Buchwald representation to construct several families of separable cylindrical solutions to the Navier-Lamé equation; these solutions had the property of being 2π -periodic in the circumferential coordinate. In this paper, we extend the analysis and obtain the complementary set of separable solutions whose circumferential parts are elementary 2π -aperiodic functions. Collectively, we construct eighteen distinct families of separable solutions; in each case, the circumferential part of the solution is one of three elementary 2π -aperiodic functions. These solutions are useful for solving a wide variety of dynamical problems that involve cylindrical geometries and for which 2π -periodicity in the angular coordinate is incompatible with the given boundary conditions. As illustrative examples, we show how the obtained solutions can be used to solve certain forced-vibration problems involving open cylindrical shells and open solid cylinders where (by virtue of the boundary conditions) 2π -periodicity in the angular coordinate is inappropriate. As an addendum to our prior work, we also include an illustrative example of a certain type of asymmetric problem that can be solved using the particular 2π -periodic subsolutions that ensue when there is no explicit dependence on the circumferential coordinate.

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1 Introduction

In linear elastodynamics [1], the displacement of a homogeneous, isotropic, and linearly-elastic solid is governed by the Navier-Lamé (NL) equation. Over the last century, many

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specialized methods have been developed for obtaining solutions to the NL equation. One of the most popular and effective methods is the method of potentials [2–4], whereby the displacement vector is represented as a specific combination of one or more scalar and/or vector functions (called potentials). The scalar components of the NL equation are a set of coupled PDEs that cannot generally be solved in closed form. Representations in terms of displacement potentials may, once substituted into the NL equation, yield a system of PDEs that are uncoupled, less coupled, or at least simpler than the original component equations. Many different representations exist including the classical Helmholtz-Lamé and Papkovitch-Neuber representations. In applications, the analytical utility of a representation often depends on the choice of the working coordinate system. One representation that has proven to be analytically effective in anisotropic problems with cylindrical symmetry is the so-called Buchwald representation (see [5–7] and references therein). The Buchwald representation is also applicable to problems involving isotropic media [8,9], but its application to such problems is relatively rare [10–12].

The Buchwald representation involves three scalar potential functions (see Eq. (2.2) of Section 2). It can be shown by various means (see, for example, [10,13]) that the Buchwald representation reduces the original scalar components of the NL equation (which are a set of three coupled PDEs) to a set of two coupled PDEs involving two of the potentials and one separate decoupled PDE involving the remaining potential. By assuming separable product solutions for the three potentials and imposing certain conditions on their axial and temporal parts, it was shown in [13] that the coupled subsystem reduces to a homogeneous linear system that can be solved by elimination. The elimination procedure generates two independent fourth-order linear PDEs with constant coefficients, which can be subsequently solved using linear combinations of solutions to the two-dimensional Helmholtz equation in polar coordinates. A solution to the independent decoupled PDE can be obtained using separation of variables. In [13] (henceforth referred to as SC1), we specifically constructed particular solutions (to the PDEs governing the Buchwald potentials) possessing 2π -periodic angular parts. In the present paper, we consider the complementary class of separable solutions whose angular parts are either aperiodic or periodic but not 2π -periodic. Although more involved, all three Buchwald potentials can again be completely determined (under the stipulated conditions), and following the approach of SC1 we can then construct eighteen distinct families of parametric solutions to the NL equation. The angular parts of these solutions (to the NL equation) are of course *not* 2π -periodic. One new complication that arises is that some solutions lacking 2π -periodicity in their angular parts are complex-valued, and in those cases, we discuss how to extract acceptable real-valued solutions.

To our knowledge, the solutions obtained here constitute the first comprehensive set of 2π -aperiodic cylindrical solutions to the NL equation. Independent of the fact that they were derived using Buchwald potentials, these exact closed-form families of solutions are fundamentally important and useful in their own right. They can be directly applied to a fundamental set of linear-elastic boundary-value-problems in cylindrical coordinates where 2π -periodicity in the angular coordinate is incompatible with the given boundary