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# A Discontinuous Galerkin Method by Patch Reconstruction for Convection-Diffusion Problems

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**Abstract.** In this article, we apply the discontinuous Galerkin method by patch reconstruction for solving convection-diffusion problems. The proposed method is highly efficient that it uses only one degree of freedom per element to achieve higher order approximation. It also enjoys the implementation flexibility on the general polygonal meshes. A priori error estimates of energy norm is devised. Numerical examples are presented to illustrate the theoretical results and the efficiency of the method.

**AMS subject classifications**: 49N45, 65N21

**Key words**: Convection-diffusion problem, polygonal mesh, discontinuous Galerkin method, patch reconstruction.

## 1 Introduction

In this article, we consider to solve the steady-state convection-diffusion equation. It is well known that the equation is hard to solve when the diffusion coefficient $\epsilon$ is small (i.e., $\epsilon \ll 1$). For the convection-dominated regime, the exact solution may possess sharp layers at the interior or the boundary. The layers are the regions where the solution has a large gradient. The standard finite element methods (FEMs) often exhibit poor stability problems, various numerical stabilization techniques have been studied for the stabilization, for a survey, we refer to [28, 29, 33] for the details. One approach is ad hoc meshing, optimize the numerical meshes via different techniques, see in [3,30,31], and the adaptive methods are also well-studied for this issue, see, e.g., [12, 13, 15]. Hughes and Brooks developed the streamline upwind/Petrov-Galerkin (SUPG) [6, 19] method which using the upwind-type scheme to stabilize the singularly perturbed problems. The upwind difference schemes for solving the convection-diffusion problems have been studied in [17,18].

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And the meshless method to solving the time-dependent problems have been proposed in [14].

During the last decades, the discontinuous Galerkin (DG) methods have been applied for various problems: see, e.g., [1] for elliptic equations, [10] for a review. The DG method for convection-diffusion problems enjoys good stability properties due to it naturally includes the upwinding which is equivalent to stabilized method [29]. Baumann and Oden proposed a nonsymmetric approximation for the steady state convection-diffusion equations in [4]. Cockburn and Shu developed the Runge-Kutta discontinuous Galerkin (RKDG) method for the convection-dominated parabolic problems in [11]. Houston et al. [16] considered the $hp$-version DG method for second-order partial differential equations with nonnegative characteristic form. Buffa et al. [7] analyzed the multiscale DG method for convection-diffusion problems which reduce the computational costs and complexity. A series of DG methods for convection-diffusion-reaction problems were analyzed in [2] by Ayuso et al. They applied the weighted-residual approach to derive the DG scheme. The derived methods have the stability and optimal error estimates in suitable norms not only for convection-dominated regimes but also diffusion-dominated, reaction-dominated and intermediate regimes. Recently, the weak Galerkin (WG) [34] methods have been developed with introduced weak gradient, it also applied to convection-diffusion problems in [8, 27]. But the above discontinuous approximation has a limitation on computational cost. Precisely, the DG method and WG method involves more degree of freedom than the conforming FEM [20, 27].

The purpose of this article is to apply the discontinuous Galerkin method by patch reconstruction for the convection-diffusion problems. The method was firstly developed to solve elliptic equations in [23], then applied to other model problems [22,25,26]. The proposed method possesses several advantages, such as the higher order approximation can be achieved with just one degree of freedoms per element. Secondly, the reconstructed approximation space actually is a sub-space of the piece-wise polynomial space commonly used by DG. Due to this fact, the DG scheme can be smoothly applied to our method. Thirdly, our method is quite friendly to polygonal meshes which will be illustrated by numerical examples.

The article is organized as follows. We describe in Section 2 for the process to construct the approximation space on the polygonal mesh and the approximation properties of such space. In Section 3, we give the weak form of the convection-diffusion problems, that is the symmetric interior penalty scheme is employed to diffusion operator discretization, for convection term, the upwinding involves. The optimal error estimate is derived in energy norms. Numerical experiments are presented in Section 4 to illustrate the method is efficient for convection-diffusion problems.

## 2 The reconstructed finite element space

We consider an open bounded Lipschitz domain $\Omega$ in $\mathbb{R}^D$, $D=2,3$, such as a convex poly-